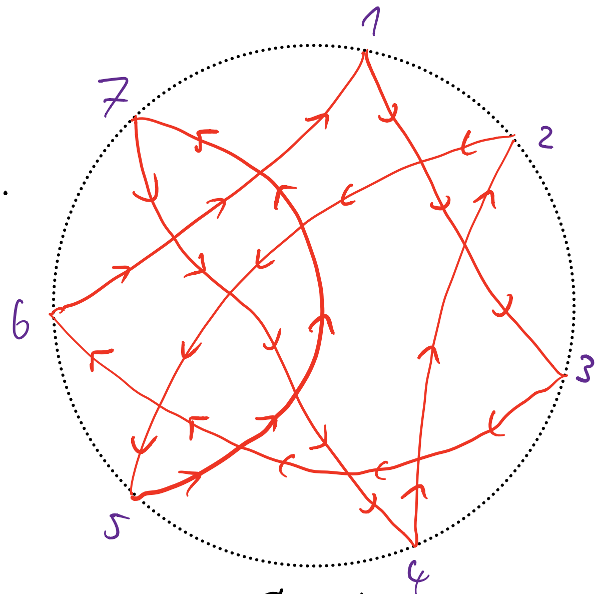
Cluster categories from Postnikov diagrams

arXiv: 1912.12475

Take oriented disc with marked pts $\{1,\dots,n\}$ in bdry.

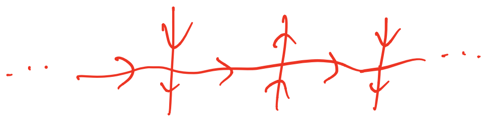
A Postnikov diagram D consists of n strands s.t.:



P0) One strand starts/ends at each marked pt.

P1) Crossings transverse, pairwise, $<\infty$ many.

P2) Signs of crossings alternate along strands



P3) Strands don't self intersect

P4) Consistency:  not allowed.

$\rightsquigarrow \sigma_D \in S_n$ by

$i \rightsquigarrow \sigma_D(i)$

k = 'average length' of strands

Say D has type (k,n).

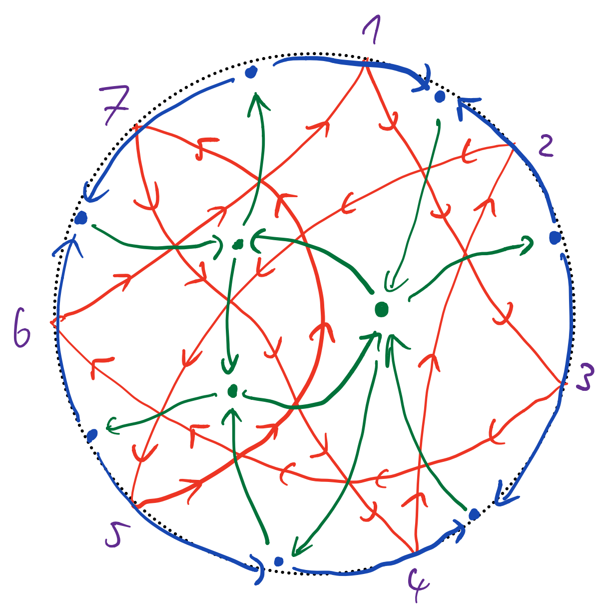
Example has type $(3,7)$.

The quiver

$D \rightsquigarrow$ quiver Q_D.

vertices $\longleftrightarrow$ alternating regions

arrows $\longleftrightarrow$ crossings



Boundary vertices (and arrows) are frozen.

$Q_D \rightsquigarrow$ cluster algebra $\mathcal{A}_D$
(with invertible froz. vars.)

<u>Thm</u> (Galashin–Lam, cf. Serhiyenko–Sherman-Bennett–Williams)

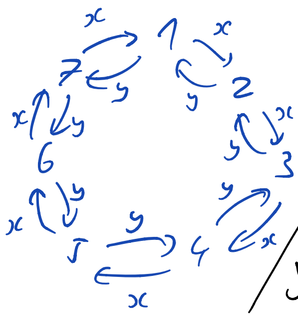
$\mathcal{A}_D \xrightarrow{\sim} \mathbb{C}[\Pi^\circ(\sigma_D)]$ for $\Pi^\circ(\sigma_D) \subseteq Gr_k^n$ open positroid variety in the Grassmannian (Postnikov). Initial cluster variables map to restr. Plücker coords.

Special case: $\sigma_D(i) = i+k \mod n \Rightarrow \Pi^\circ(\sigma_D) \subseteq Gr_n^{\wedge}$ is dense.

In this case theorem is due to Scott.

For this case A_D has a categorification $CM(C(k,n))$ by Jensen-King-Su:

- stably 2-CY Frobenius category
- (reachable) rigid objects $\Leftrightarrow$ cluster monomials, indec. projectives $\Leftrightarrow$ frozen vars.
- (reachable) cluster-tilting objects $\Leftrightarrow$ clusters
- mutation $\Leftrightarrow$ mutation etc.

$C(k,n) =$ 

$(n=7)$

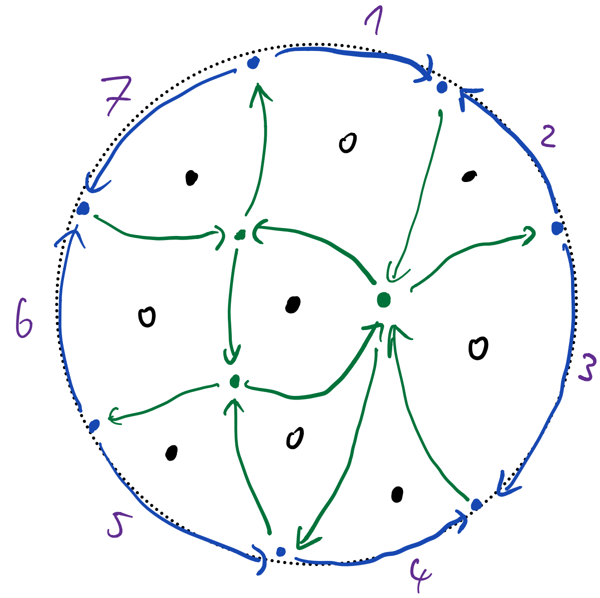
$\Big/ \begin{matrix} xy = yx \\ y^k = x^{n-k} \end{matrix}$

<u>Aim</u> Categorify in general.

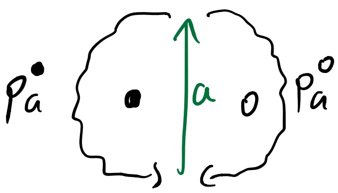
<u>The dimer algebra</u>

$D \rightsquigarrow A_D$ dimer algebra

Q has distinguished set of anticlockwise ($\bullet$) and clockwise ($\circ$) cycles.



$A_D = \widehat{\mathbb{C}Q_D} / \text{relations } p_a^\bullet = p_a^\circ$



Let $e = e^2 \in A_D$ be sum of boundary idempotents, $B := eA_De$ (boundary algebra).

Our categorification is $GP(B) := \{X \in \operatorname{mod} B : \operatorname{Ext}_B^{i>0}(X,B) = 0\}$.

Precisely:

<u>Thm</u> (P) Let D be a connected Postnikov diagram with dimer algebra A_D, boundary algebra B. Then:

1) B is ≤ 3-Iwanaga-Gorenstein, i.e. Noetherian, $\operatorname{injdim}_B B$, $\operatorname{injdim} B_B \leq 3$.

$\Rightarrow GP(B)$ is a Frobenius category

2) Stable category $\underline{GP}(B) = GP(B)/\mathrm{proj}\, B$ is 2-Calabi-Yau triangulated.

3) $A_D = \mathrm{End}_B(eA_D)^{op}$ and $eA_D \in GP(B)$ is cluster-tilting, i.e.

$$\mathrm{add}(eA_D) = \{X \in GP(B) : \mathrm{Ext}^1_B(X, eA) = 0\}.$$

Follows from general result, using properties of A_D:

1) Noetherian 2) $\dim A_D/A_DeA_D < \infty$ 3) A_D internally 3-CY w.r.t. e.

A internally 3CY w.r.t. $e \Rightarrow \mathrm{Ext}^i_A(X,Y) = \mathrm{Ext}^{3-i}_A(Y,X)^*$ for $X, Y \in \mathrm{mod}\, A$, $eY = 0$.
and $\mathrm{gldim}\, A \leq 3$.

(Def is slightly stronger, technical.)

Lemma (Çanakçı-King-P) D connected $\Rightarrow A_D$ has central subalgebra $Z \cong \mathbb{C}[\![t]\!]$,
and $e_jA_De_i \cong Z$ $\forall i, j \in Q_0$.

Lemma $\Rightarrow$ (1), (2) directly, and plays a role in proof of (3).

Remarks 1) Like $A_D \xrightarrow{\sim} \mathbb{C}[\Pi^0(\sigma_D)]$, B depends only on σ_D.

2) A_D can be equivalently defined from a consistent dimer model in the disc. Int. 3-CY property analogous to 3-CY property for algebras from consistent dimer models in the torus (Broomhead).

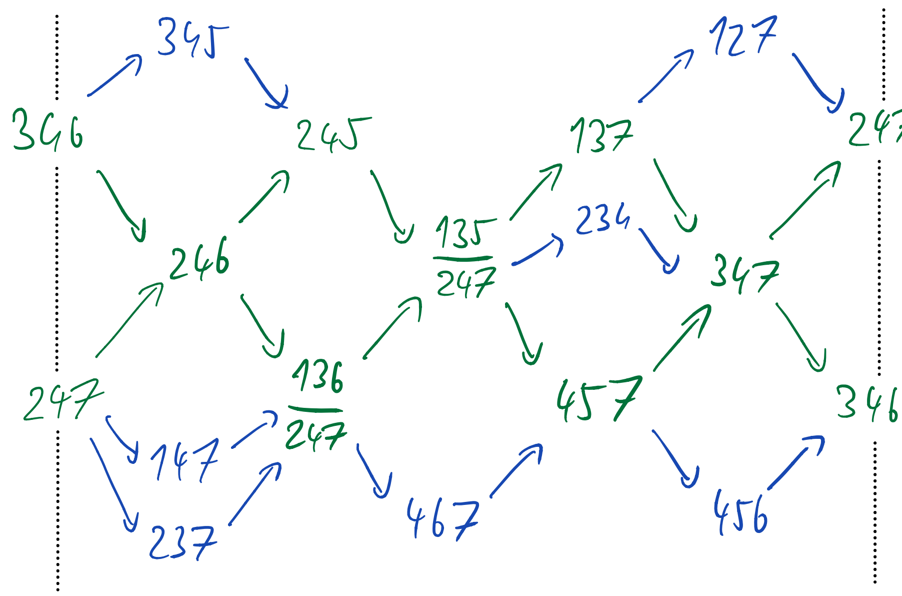
3) If $\sigma_D(i) = i+k \mod n$, then $B \cong C(k,n)$ (Baur-King-Marsh) and $GP(B) = CM(B) \rightsquigarrow$ recover JKS category

Relationship to JKS category

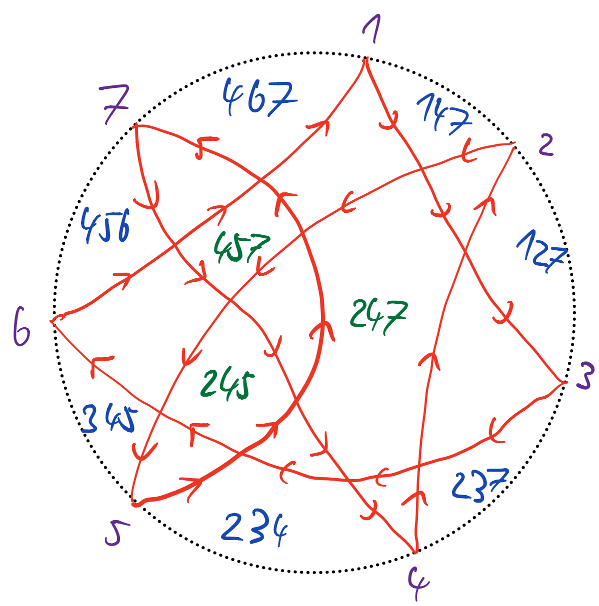
Prop (Çanakçı-King-P) If D has type (k,n), there is a canonical ring morphism $C(k,n) \to B$, inducing a fully faithful functor $\rho: GP(B) \to CM(C(k,n))$.

⇝ Our categorifications embed in the appropriate JKS category.

Example For running example D, $GP(B) \subseteq CM(3,7)$ is



Labelling



Label alternating region (quiver vertex) by i if it is left of strand starting at i

⇝ $I_j \subseteq \{1,\dots,n\}$ $\forall j \in Q_0$, $|I_j| = k$.

$A_D \xrightarrow{\sim} \mathbb{C}[\Pi^\circ(\sigma_D)]$

$x_j \mapsto \Delta(I_j)|_{\Pi^\circ(\sigma_D)}$ Plücker coordinate

JKS category $CM(C(k,n))$ has rank 1 indecomposable rigid object M_I $\forall I \subseteq \{1,\dots,n\}$, $|I| = k$.

Thm (Çanakçı-King-P) $\forall j \in Q_0$, $p(eA_De_j) \cong M_{I_j}$.

+ earlier results $\Rightarrow$ $A_D \cong \mathrm{End}_{C(k,n)}\left(\bigoplus_{j \in Q_0} M_{I_j}\right)^{op}$. (cf. Baur-King-Marsh)