Graded Frobenius cluster categories

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Graded cluster algebras

- Recall that a cluster algebra is, in particular, an algebra with a distinguished set of generators (cluster variables).
- A ($\mathbb{Z}$-)grading of a cluster algebra is a grading of the underlying algebra such that all the cluster variables are homogeneous.
- Cluster algebras are 'generated' by seeds $(\underline{x}, B)$, where $\underline{x} = (x_1, \ldots, x_r, x_{r+1}, \ldots, x_n)$ and B is an $n \times r$ integer 'exchange' matrix. (The last $n - r$ entries of $\underline{x}$ are 'frozen'.)
- We can specify a grading locally via $G \in \mathbb{Z}^n$ such that

$$B^t G = 0,$$

  by $\deg(x_i) = G_i$.
- This compatibility condition ensures that the exchange relations

$$x_k x_k' = \prod_{b_{ik}>0} x_i^{b_{ik}} + \prod_{b_{ik}<0} x_i^{-b_{ik}}$$

  are homogeneous. Then $\deg(x_k') = \sum_{b_{ik}>0} G_i - G_k =: G_k'$, and we can propagate via mutation.

Cluster categories

- A cluster category is, in particular, a 2-Calabi–Yau triangulated category with cluster-tilting objects.
- Such categories model the combinatorics of cluster algebras, with cluster-tilting objects replacing the seeds.
- Let $\mathcal{C}$ be a cluster category, and let $T = \bigoplus_{i=1}^{r} T_i \in \mathcal{C}$ be a cluster-tilting object. Write $\Lambda = \mathrm{End}_{\mathcal{C}}(T)^{\mathrm{op}}$.
- Let $F, G \colon \mathcal{C} \to \operatorname{mod} \Lambda$ be given by $F = \mathrm{Hom}_{\mathcal{C}}(T, -)$ and $G = \mathrm{Ext}^1_{\mathcal{C}}(T, -) := \mathrm{Hom}_{\mathcal{C}}(T, \Sigma -)$ respectively.
- The cluster-tilting object T has mutations $\mu_k T = T/T_k \oplus T_k'$ for each $1 \leq k \leq r$, where T_k' is determined via exchange triangles

  $$T_k \to X_k \to T_k' \to \qquad T_k' \to Y_k \to T_k \to$$

  with X_k and Y_k in $\operatorname{add} T$ (provided the quiver of Λ has no loops or 2-cycles at k).
- This models the mutation of seeds in a cluster algebra, with the exchange relations corresponding to these exchange triangles.

Cluster characters

- Let $X \in \mathcal{C}$ be any object. Since T is cluster-tilting, there is a distinguished triangle $T^{m(X)} \to T^{p(X)} \to X \to$ in $\mathcal{C}$, where $m(X), p(X) \in \mathbb{Z}^r$, and $T^v := \bigoplus_{i=1}^r T_i^{v_i}$, such that

$$FT^{m(X)} \to FT^{p(X)} \to FX \to 0$$

  is exact in $\operatorname{mod}\Lambda$. Write $\operatorname{ind}_T(X) = p(X) - m(X)$.
- Let B_T have (i,j)-th entry $\dim \operatorname{Ext}^1_\Lambda(S_i, S_j) - \dim \operatorname{Ext}^1_\Lambda(S_j, S_i)$, where the $S_k = \operatorname{top} FT_k$ are the simple Λ-modules.
- The cluster character of X with respect to T is

$$\varphi_X^T = \underline{x}^{\operatorname{ind}_T(X)} \sum_{v \in \mathbb{Z}^r} \lambda_v \underline{x}^{B_T v} \in \mathbb{C}[x_1^{\pm}, \dots, x_r^{\pm}],$$

  where λ_v 'counts' the number of dimension v submodules of GX (so $\lambda_v = 0$ unless, componentwise, $0 \leq v \leq \underline{\dim}\, GX$).
- In nice cases, the φ_X^T for X rigid and indecomposable are then the cluster variables of the cluster algebra generated by $(\underline{x}, B_T)$, which we say is categorified by $\mathcal{C}$ (or by the pair $(\mathcal{C}, T)$).

Graded cluster categories

- A grading of $\mathcal{C}$ (with respect to T) is $G \in \mathbb{Z}^r$ such that $B_T^t G = 0$.
- For $X \in \mathcal{C}$, define $\deg_G(X) = \operatorname{ind}_T(X) \cdot G$.

Proposition (Grabowski '15)

(i) $\deg_G(X) = \deg(\varphi_X^T) \in \mathbb{C}[x_1^{\pm}, \ldots, x_r^{\pm}]$, *where* $\deg(x_i) = G_i$,

(ii) $\deg_G(Y) = \deg_G(X) + \deg_G(Z)$ *whenever* $X \to Y \to Z \to$ *is a distinguished triangle,*

(iii) *if* $T_k \to X_k \to T_k' \to$ *and* $T_k' \to Y_k \to T_k \to$ *are exchange triangles, then* $\deg(X_k) = \deg(T_k) + \deg(T_k') = \deg(Y_k)$, *and*

(iv) *for all* $X \in \mathcal{C}$, $\deg_G(X) = -\deg_G(\Sigma X)$.

- Parts (iii) and (iv) are simple consequences of (ii).
- Part (iv) shows that objects of degree d are in bijection with those of degree $-d$. This property translates to cluster algebras categorified by a triangulated category $\mathcal{C}$ (in the sense above, so that cluster variables correspond to *all* indecomposable rigid objects).

A global definition

- Unlike the original algebraic definition of grading, this categorical version is ‘local’, relying on a choice of cluster-tilting object.

Proposition (Grabowski ’15)

(v) *The space of gradings for* $\mathcal{C}$ *is isomorphic to* $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_0(\mathcal{C}), \mathbb{Z})$, *via* $G \mapsto \deg_G$.

- This gives a global definition of a grading, equivalent to the local one in terms of T.
- It also tells us how to write the same grading in terms of any cluster-tilting object of $\mathcal{C}$, irrespective of whether it can be obtained from T by a finite sequence of mutations.
- The proof is essentially by rephrasing a result of Palu (’09), who gives a presentation of $\mathrm{K}_0(\mathcal{C})$ in terms of the generators $[T_i]$, in the language of gradings.

Frobenius cluster categories

- A cluster algebra categorified by a cluster category as above necessarily has square exchange matrices, so there are no frozen variables. This does not apply to many cluster algebras in nature.
- A Frobenius category is an exact category $\mathcal{E}$ with enough projective and injective objects, which coincide.
- The indecomposable projective-injective objects appear as summands of every cluster-tilting object, and play the role of the frozen variables. Factoring out maps through these objects produces a triangulated category $\underline{\mathcal{E}}$.

Definition

A Frobenius category $\mathcal{E}$ is a Frobenius cluster category if it is Krull–Schmidt, $\underline{\mathcal{E}}$ is 2-Calabi–Yau, it has cluster-tilting objects, and every such object T satisfies $\operatorname{gl.dim}\operatorname{End}_{\mathcal{E}}(T)^{\mathrm{op}} \leq 3$.

- While $\mathcal{E}$ need not be Hom-finite, $\underline{\mathcal{E}}$ must be, as this is part of the definition of 2-Calabi–Yau.

Notes on assumptions

- In the triangulated case, the assumption that $\operatorname{gl.dim}\operatorname{End}_{\mathcal{E}}(T)^{\mathrm{op}} \leq 3$ for all cluster-tilting objects would be totally unreasonable, and exclude almost all examples. In the Frobenius case it is much more benign.
- Many examples of such categories are described in Buan–Iyama–Reiten–Scott '09 and several papers of Geiß–Leclerc–Schröer, and we will see another family later.
- Pick a cluster-tilting object $T = \bigoplus_{i=1}^{n} T_i \in \mathcal{E}$, and write $\Lambda = \operatorname{End}_{\mathcal{E}}(T)^{\mathrm{op}}$, $F = \operatorname{Hom}_{\mathcal{E}}(T, -)$ and $G = \operatorname{Ext}^1_{\mathcal{E}}(T, -)$. We number summands so that T_i is non-projective if and only if $i \leq r$.
- The assumption that $\mathcal{E}$ is Krull–Schmidt means that $\Lambda = \operatorname{End}_{\mathcal{E}}(T)^{\mathrm{op}}$ has a complete set of indecomposable projectives given by $P_i = FT_i$, whose simple tops S_i are a complete set of simples.
- Since $\underline{\mathcal{E}}$ is 2-Calabi–Yau, essentially the same statements about mutations and indices work as before, but with triangles replaced by short exact sequences. One can only mutate the non-projective indecomposable summands of T, i.e. those which are indecomposable (i.e. non-zero) in $\underline{\mathcal{E}}$.

Grothendieck groups and the Euler form

- We will also assume that Λ is Noetherian; then the Grothendieck group $\mathrm{K}_0(\operatorname{mod}\Lambda)$ has basis $[P_i]$, dual to the basis $[S_i]$ of $\mathrm{K}_0(\operatorname{fd}\Lambda)$ under the Euler form

$$\langle M, N\rangle = \sum_{i=0}^{3}(-1)^i \dim \operatorname{Ext}^i_\Lambda(M,N).$$

- Using the Euler form, we can write the standard cluster character (with respect to T) on $\mathcal{E}$ as

$$\varphi^T_X = \prod_{i=1}^{n} x_i^{\langle FX, S_i\rangle} \sum_{v\in\mathbb{Z}^r} \lambda_v \prod_{i=1}^{n} x_i^{-\langle v, S_i\rangle} \in \mathbb{C}[x_1^{\pm}, \dots, x_n^{\pm}].$$

- This is implied by a more general formula [Fu–Keller '10], using that $\operatorname{p.dim} FX \leq 1$ for all X, that Λ is 'internally 3-Calabi–Yau' [Keller–Reiten '07, P '15], and that $\langle M, S_i\rangle$ depends only on $v = \underline{\dim}\, M$ when M is a submodule of GX [Fu–Keller '10].

Graded Frobenius cluster categories

- The algebra Λ has a canonical quotient $\underline{\Lambda} = \mathrm{End}_{\underline{\mathcal{E}}}(T)^{\mathrm{op}}$ given by factoring out maps through the projective summands of T. The simple $\underline{\Lambda}$-modules are S_i for $i \leq r$.
- Since $\underline{\Lambda}$ is finite-dimensional, the class of any $\underline{\Lambda}$-module in $\mathrm{K}_0(\mathrm{mod}\,\Lambda)$ lies in the span of these r simple modules.
- A grading for the Frobenius cluster category $\mathcal{E}$ is $G \in \mathrm{K}_0(\mathrm{fd}\,\Lambda)$ such that
$$\langle M, G \rangle = 0$$
for all $M \in \mathrm{mod}\,\underline{\Lambda}$.
- As before, let B_T have entries
$$(B_T)_{ij} = \dim \mathrm{Ext}^1_\Lambda(S_i, S_j) - \dim \mathrm{Ext}^1_\Lambda(S_j, S_i)$$
for $1 \leq i \leq n$ and $1 \leq j \leq r$. Then, again by internal Calabi–Yau symmetry, we have
$$(B_T)_{ji} = -\langle S_j, S_i \rangle,$$
so G is a grading if and only if $B_T^t G = 0$ (writing G in the basis of simples).

Basic properties

- For any $X \in \mathcal{E}$, let $\deg_G(X) = \langle FX, G \rangle$.
- As with the compatibility condition, this can be written as $\deg_G(X) = \operatorname{ind}_T(X) \cdot G$, where $\operatorname{ind}_T(X) = p(X) - m(X)$ is defined by the existence of an exact sequence

$$0 \to T^{m(X)} \to T^{p(X)} \to X \to 0.$$

- However, the equivalent 'coordinate-free' definitions using the Grothendieck groups of Λ are better adapted to our arguments.

Proposition (GP '16)

(i) $\deg_G(X) = \deg(\varphi_X^T)$, *where* $\deg(x_i) = \langle P_i, G \rangle$, *or equivalently* $G = \sum_{i=1}^n \deg(x_i)[S_i]$ *when expressed in the basis of simples,*

(ii) $\deg_G(Y) = \deg_G(X) + \deg_G(Z)$ *whenever* $0 \to X \to Y \to Z \to 0$ *is an exact sequence, and*

(iii) *if* $0 \to T_k \to X_k \to T'_k \to 0$ *and* $0 \to T'_k \to Y_k \to T_k \to 0$ *are exchange sequences, then* $\deg(X_k) = \deg(T_k) + \deg(T'_k) = \deg(Y_k)$.

A global definition

- We again have a statement linking gradings to the Grothendieck group of the categorification.

Theorem (GP '16)

The space of gradings for $\mathcal{E}$ is isomorphic to $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}_0(\mathcal{E}), \mathbb{Z})$, via $G \mapsto \deg_G$.

- Again we obtain a global definition, allowing us to write any grading in terms of an arbitrary cluster-tilting object.
- The proof again uses ideas of Palu, but is more than just a translation. Palu gives an exact sequence

$$\mathrm{K}_0(\mathcal{H}^{\mathrm{b}}_{\mathcal{E}\text{-ac}}(\operatorname{add} T)) \to \mathrm{K}_0(\mathcal{H}^{\mathrm{b}}(\operatorname{add} T)) \to \mathrm{K}_0(\mathcal{D}^{\mathrm{b}}(\mathcal{E})) \to 0$$

which we show is isomorphic to

$$\mathrm{K}_0(\operatorname{mod} \underline{\Lambda}) \xrightarrow{\psi} \mathrm{K}_0(\operatorname{mod} \Lambda) \to \mathrm{K}_0(\mathcal{E}) \to 0.$$

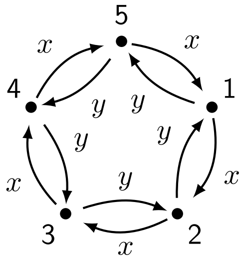
The claim follows by computing $\mathrm{Hom}_{\mathbb{Z}}(\psi, \mathbb{Z})$ explicitly enough to see that its kernel is the space of gradings.

Examples of gradings

- One powerful feature of this theorem is that it allows us to check that some piece of homological data is a grading by checking that it is additive on exact sequences, which is typically much easier than checking compatibility with an exchange matrix.
- Conversely, it explains how to use an exchange matrix to get an easy computation of the rank of the Grothendieck group.
- As an example, let $\mathcal{E}$ be Hom-finite, and let $P \in \mathcal{E}$ be projective-injective. Then $\dim \operatorname{Hom}_{\mathcal{E}}(P, -)$ and $\dim \operatorname{Hom}_{\mathcal{E}}(-, P)$ both define gradings, since the functors involved are exact.
- If $\mathcal{E} \subseteq \operatorname{mod} \Pi$, with the inherited exact structure, then the dimension vectors of objects of $\mathcal{E}$ as Π-modules give (multi-)gradings.
- Warning: under our assumptions (including Noetherianity of Λ), any Frobenius cluster category embeds into a module category as above [Iyama–Kalck–Wemyss–Yang '14]. But these embeddings are not unique, and so one can potentially treat objects of $\mathcal{E}$ as modules over different algebras, resulting in different dimension vectors.

Grassmannian cluster categories

- A particularly interesting class of cluster algebras are the cluster structures on the coordinate rings of Grassmannians G_k^n of k-planes in $\mathbb{C}^n$ [Scott '06], in which all Plücker coordinates appear as cluster variables (but there are usually more).
- These structures have been categorified [Jensen–King–Su '16] by categories $\mathrm{CM}(A)$, where A is the completed path algebra of the quiver



(drawn for $n = 5$) modulo relations $xy = yx$ and $x^k = y^{n-k}$. The centre of A is $Z = \mathbb{C}[[t]]$ for $t = xy$.

The rank of a module

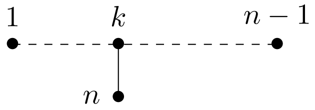
- One can show [P '15] that $\mathrm{CM}(A)$ is a Frobenius cluster category, and the endomorphism algebras of its cluster-tilting objects are Noetherian (but not finite-dimensional).
- An A-module is Cohen–Macaulay if and only if it is free and finitely generated as a Z-module. In particular, each object $X \in \mathrm{CM}(A)$ has a well-defined integer rank as a Z-module, and such ranks are additive on exact sequences, thus giving a grading.
- Taken literally, these ranks are always multiples of n, so we define $\mathrm{rk}\, X$ by dividing out n from the 'honest' rank.
- The corresponding grading on the cluster algebra, which is the coordinate ring of the Grassmannian and thus generated by Plücker coordinates, is given by the degree of an element as a polynomial in these coordinates.
- In particular, there are precisely $\binom{n}{k}$ degree 1 cluster variables, which are the Plücker coordinates themselves.

The Grothendieck group

- Jensen–King–Su calculate the Grothendieck group of $\mathrm{CM}(A)$, showing that it is isomorphic to

$$\mathbb{Z}^n(k) = \Big\{v \in \mathbb{Z}^n : \sum_{i=1}^{n} v_i \in k\mathbb{Z}\Big\}.$$

- This lattice may also be realised as the root lattice of the Kac–Moody Lie algebra associated to the graph $J_{k,n}$:



- A basis of simple roots is

$$\alpha_i = e_{i+1} - e_i,\ 1 \leq i \leq n-1 \qquad \beta_{[n]} = e_1 + \cdots + e_k,$$

and under the isomorphism of $\mathbb{Z}^n(k)$ with $\mathrm{K}_0(\mathrm{CM}(A))$, the function rk corresponds to the function giving the $\beta_{[n]}$-coordinate.

Open questions

- There are still many unanswered questions about the grading of a general Grassmannian cluster algebra by Plücker degree, such as:
  - are the degrees of cluster variables unbounded?
  - does every integer value appear as a degree?
  - how many cluster variables are there in each degree?

  although these are beginning to be addressed [Booker-Price].
- In finite types $(k,n) \in \{(1,n),(2,n),(3,6),(3,7),(3,8)\}$, i.e. when the graph $J_{k,n}$ is a Dynkin diagram, the number of cluster variables of degree d is d times the number of $J_{k,n}$-roots with $\beta_{[n]}$-coefficient d.
- Since in these cases there are finitely many cluster variables, their degrees must be bounded; the maximal degrees are 1, 1, 2, 2 and 3 respectively.
- The formula in terms of $J_{k,n}$-roots does not hold in infinite type however; in the case $(3,9)$ there are more degree 3 cluster variables than this root system predicts.
- We hope that having a categorical interpretation of the grading may open these problems up to attack by representation theoretic methods.