# Internally Calabi–Yau Algebras

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# Main Definition

- Let $A$ be a (not necessarily finite dimensional) $\mathbb{C}$-algebra, and let $e$ be an idempotent of $A$.
- Throughout, we will write $\underline{A} = A/AeA$ (the interior algebra) and $B = eAe$ (the boundary algebra).

## Definition

The algebra $A$ is internally $d$-Calabi–Yau with respect to $e$ if

(i) $\operatorname{gl.dim} A \leq d$, and

(ii) for any finite dimensional $\underline{A}$-module $M$, and any $A$ module $N$, there is a duality

$$\mathrm{D}\operatorname{Ext}^i_A(M, N) = \operatorname{Ext}^{d-i}_A(N, M)$$

for all $i$, functorial in $M$ and $N$.

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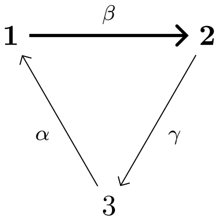
- Setting $e = 0$ recovers the (naïve) definition of a $d$-Calabi–Yau algebra.
- Setting $e = 1$, (ii) becomes vacuous.
- If $e \neq 1$, (ii) $\implies \operatorname{gl.dim} A \geq d$, and so $\operatorname{gl.dim} A = d$ in this case.

# Example 1 (finite dimensional, $d = 3$)

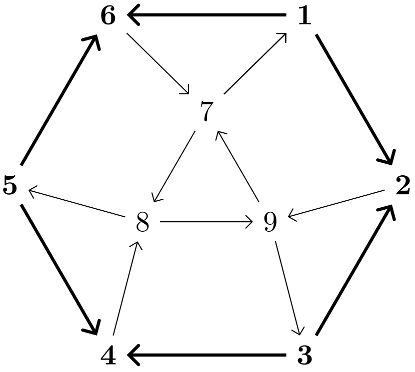



$$\beta\alpha = 0 = \gamma\beta$$

$$e = e_1 + e_2$$

- $\underline{A} = \mathbb{C}$.
- $B = eAe$ is the preprojective algebra of type $\mathsf{A}_2$.

# Example 2 (infinite dimensional, $d = 3$)




The two paths back along any internal arrow are equal.

$$e = \sum_{i=1}^{6} e_i$$

# Origins

- Let $\mathcal{E}$ be a Frobenius category: an exact category with enough projectives and enough injectives, and such that projective and injective objects coincide.
- Then $\underline{\mathcal{E}} = \mathcal{E}/\operatorname{proj}\mathcal{E}$ is triangulated.
- Assume that $\mathcal{E}$ is Krull–Schmidt, and $\underline{\mathcal{E}}$ is $d$-Calabi–Yau.
- Let $T \in \mathcal{E}$ be $d$-cluster-tilting, i.e.

$$\operatorname{add} T = \{X \in \mathcal{E} : \operatorname{Ext}^i_{\mathcal{E}}(X, T) = 0,\ 0 < i < d\}.$$

**Theorem (Keller–Reiten)**

*If $\operatorname{gl.dim}\operatorname{End}_{\mathcal{E}}(T)^{\mathrm{op}} \leq d+1$, then it is internally $(d+1)$-Calabi–Yau with respect to projection onto a maximal projective summand.*

# Bimodule version

- Write $A^\varepsilon = A \otimes_{\mathbb{C}} A^{\mathrm{op}}$, and $\Omega_A = \mathrm{RHom}_{A^\varepsilon}(A, A^\varepsilon)$. Let $\mathcal{D}_{\underline{A}}(A)$ be the full subcategory of the derived category of $A$ consisting of objects whose total cohomology is a finite-dimensional $\underline{A}$-module.

## Definition

The algebra $A$ is internally bimodule $d$-Calabi–Yau with respect to $e$ if

(i) $\mathrm{p.dim}_{A^\varepsilon} A \leq d$, and

(ii) there is a triangle
$$A \to \Omega_A[d] \to C \to A[1]$$
in $\mathcal{D}(A^\varepsilon)$, such that $\mathrm{RHom}_A(C, M) = 0 = \mathrm{RHom}_{A^{\mathrm{op}}}(C, N)$ for all $M \in \mathcal{D}_{\underline{A}}(A)$ and $N \in \mathcal{D}_{\underline{A}^{\mathrm{op}}}(A^{\mathrm{op}})$.

- If we can take $C = 0$, then $A \cong \Omega_A[d] \in \mathrm{per}\, A^\varepsilon$ is bimodule $d$-Calabi–Yau.

# Consequences

## Definition

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- $A$ is internally bimodule $d$-Calabi–Yau with respect to $e$ if and only if the same is true for $A^{\mathrm{op}}$.
- If $A$ is internally bimodule $d$-Calabi–Yau with respect to $e$ then

$$\mathrm{D}\operatorname{Hom}_{\mathcal{D}(A)}(M, N) = \operatorname{Hom}_{\mathcal{D}(A)}(N, M[d])$$

for any $N \in \mathcal{D}(A)$ and any $M \in \mathcal{D}_{\underline{A}}(A)$.

- In particular, such an $A$ is internally $d$-Calabi–Yau with respect to $e$.

# Main Theorem

## Theorem (P, cf. Amiot–Iyama–Reiten)

*Let $A$ be a Noetherian algebra, and $e$ an idempotent such that $\underline{A}$ is finite dimensional. If $A$ and $A^{\mathrm{op}}$ are internally $(d+1)$-Calabi–Yau with respect to $e$, then*

(i) *$B$ is Iwanaga–Gorenstein of Gorenstein dimension at most $d+1$, and so*

$$\mathrm{GP}(B) = \{X \in \operatorname{mod} B : \operatorname{Ext}^i_B(X, B) = 0,\ i > 0\}$$

*is Frobenius,*

(ii) *$eA \in \mathrm{GP}(B)$ is $d$-cluster-tilting, and*

(iii) *there are natural isomorphisms $A \cong \operatorname{End}_B(eA)^{\mathrm{op}}$ and $\underline{A} \cong \operatorname{End}_{\underline{\mathrm{GP}}(B)}(eA)^{\mathrm{op}}$.*

*If $A$ is internally bimodule $(d+1)$-Calabi–Yau with respect to $e$, then additonally*

(iv) *$\underline{\mathrm{GP}}(B)$ is $d$-Calabi–Yau.*

# Frozen Jacobian algebras

- Let $Q$ be a quiver, and $F$ a (not necessarily full) subquiver, called frozen.
- Let $W$ be a linear combination of cycles of $Q$.
- For a cyclic path $\alpha_n \cdots \alpha_1$ of $Q$, define

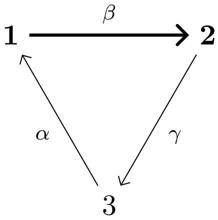
$$\partial_\alpha(\alpha_n \cdots \alpha_1) = \sum_{\alpha_i = \alpha} \alpha_{i-1} \cdots \alpha_1 \alpha_n \cdots \alpha_{i+1}$$

  and extend by linearity.
- The frozen Jacobian algebra $J(Q, F, W)$ is

$$J(Q, F, W) = \mathbb{C}Q / \langle \partial_\alpha W : \alpha \in Q_1 \setminus F_1 \rangle,$$

  where $\mathbb{C}Q$ denotes the complete path algebra of $Q$ over $\mathbb{C}$.
- The frozen idempotent is $e = \sum_{i \in F_0} e_i$.

# Example




$F$ is the full subquiver on vertices 1 and 2.

$$W = \gamma\beta\alpha$$

$$e = e_1 + e_2$$

# A bimodule resolution?

- Let $A$ be a frozen Jacobian algebra, let $S = A/\mathfrak{m}(A)$ be the semisimple part of $A$, and write $\otimes = \otimes_S$. Write $\overline{Q}_i^{\mathrm{m}}$ for the dual $S$-bimodule to $Q_i \setminus F_i$.
- There is a natural complex

$$0 \to A \otimes \overline{Q}_0^{\mathrm{m}} \otimes A \to A \otimes \overline{Q}_1^{\mathrm{m}} \otimes A \to A \otimes Q_1 \otimes A \to A \otimes Q_0 \otimes A \to A \to 0$$

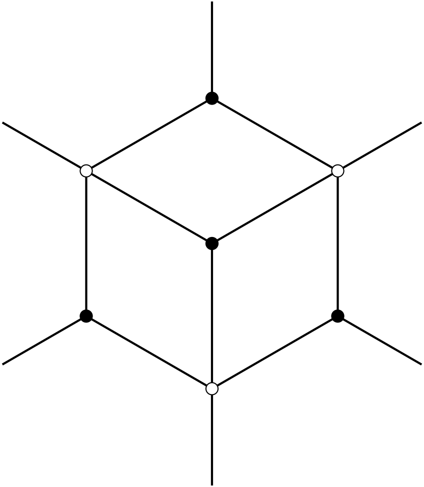
of $A$-bimodules (cf. Ginzburg and Broomhead for the case $F = \varnothing$).

**Theorem (P)**

*If this complex is exact, then $A$ is internally bimodule $3$-Calabi–Yau with respect to the frozen idempotent $e$.*

# Dimer models

Definition by example (in the disk):



# Associated frozen Jacobian algebra

Definition by example (in the disk):