From frieze patterns to cluster categories

Matthew Pressland

University of Leeds

LMS Autumn Algebra School

Rough plan

Lecture I: Frieze patterns
Conway and Coxeter, early 70s.

Lecture II: Cluster algebras
Fomin and Zelevinsky, early 2000s.

Lecture III: Cluster categories
Buan, Marsh, Reineke, Reiten and Todorov, 2006.

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Part I: Frieze patterns

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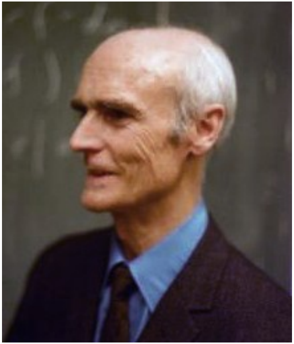
Conway and Coxeter

John H. Conway
Photo: Thane Plambeck



H.S.M. Coxeter
Photo: Konrad Jacobs, Erlangen

Frieze patterns (Conway–Coxeter)

A *frieze pattern* of *height* n consists of $n+2$ rows of positive integers

$$
\begin{array}{ccccccccccccccccccccccc}
\cdots & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \cdots \\
 & 3 & & 1 & & 2 & & 3 & & 2 & & 2 & & 2 & & 1 & & 5 & & 3 & & 1 & \\
\cdots & & 2 & & 1 & & 5 & & 5 & & 3 & & 3 & & 1 & & 4 & & 14 & & 2 & & \cdots \\
 & 9 & & 1 & & 2 & & 8 & & 7 & & 4 & & 1 & & 3 & & 11 & & 9 & & 1 & \\
\cdots & & 4 & & 1 & & 3 & & 11 & & 9 & & 1 & & 2 & & 8 & & 7 & & 4 & & \cdots \\
 & 3 & & 3 & & 1 & & 4 & & 14 & & 2 & & 1 & & 5 & & 5 & & 3 & & 3 & \\
\cdots & & 2 & & 2 & & 1 & & 5 & & 3 & & 1 & & 2 & & 3 & & 2 & & 2 & & \cdots \\
 & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} &
\end{array}
$$

such that

(1) every entry in the first and final row is 1, and

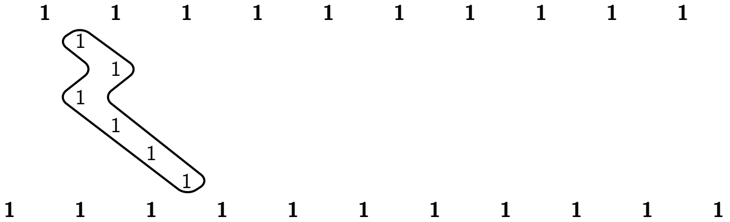
(2) the entries satisfy the SL_2 *diamond rule*, meaning that every local configuration $\begin{smallmatrix} & b & \\ a & & d \\ & c & \end{smallmatrix}$ satisfies $ad - bc = 1$.

We call the first and last row of the frieze, consisting only of 1s, *trivial* rows. The height measures the number of non-trivial rows.

Lightning bolts

Because of the SL_2 diamond rule, we can compute friezes recursively from appropriate initial conditions.

In these lectures, we are most interested in starting with the entries of a lightning bolt: one entry per row, with entries in successive rows in the same diamond.



Starting from the values in a lightning bolt, we can compute all entries, but this requires division

$$\begin{smallmatrix} & b & \\ a & & d \\ & c & \end{smallmatrix} \implies d = \frac{1+bc}{a}$$

so we need not obtain integers as we require.

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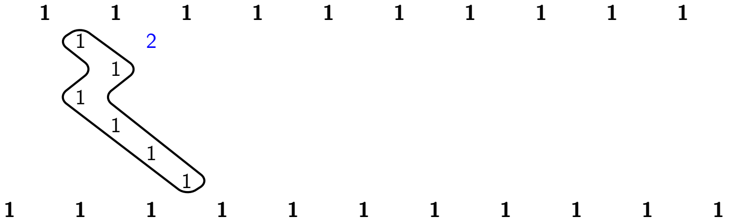

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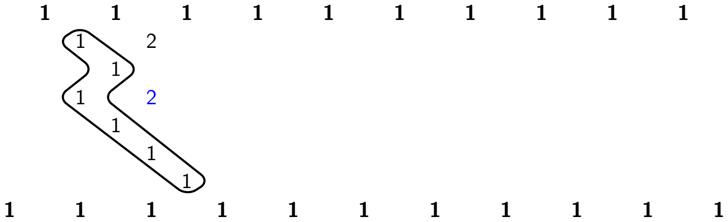

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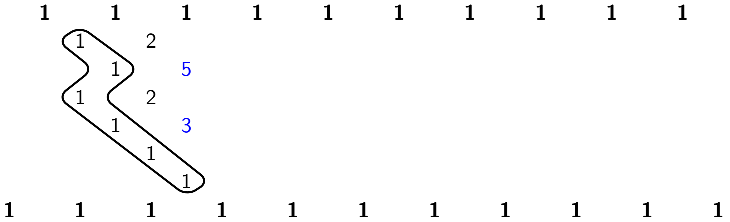

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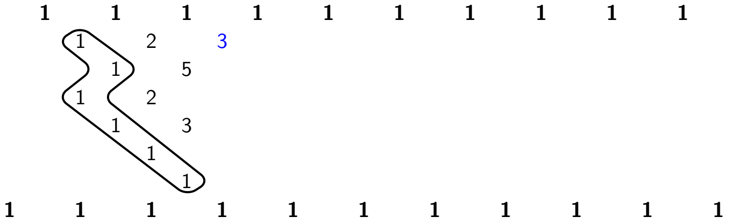

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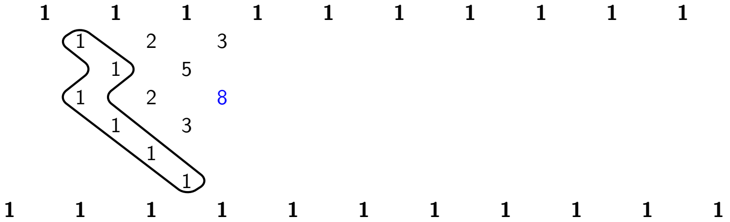

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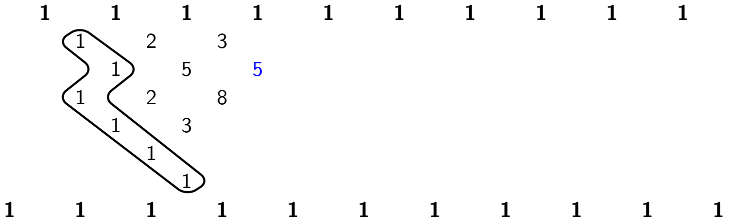

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⋯		**1**		**1**		**1**		**1**		**1**		**1**		**1**		**1**		**1**		**1**		⋯
	3		1		2		3		2		2		2		1		5		3		1	
⋯		2		1		5		5		3		3		1		4		14		2		⋯
	9		1		2		8		7		4		1		3		11		9		1	
⋯		4		1		3		11		9		1		2		8		7		4		⋯
	3		3		1		4		14		2		1		5		5		3		3	
⋯		2		2		1		5		3		1		2		3		2		2		⋯
	1		**1**		**1**		**1**		**1**		**1**		**1**		**1**		**1**		**1**		**1**	

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Integrality

Phenomenon (Integrality)

Given a lightning bolt, setting its entries equal to 1 determines a unique frieze pattern. More concretely, we can compute all other entries via the diamond rule, and all of them are positive integers, as the definition requires.

We will explain this phenomenon via cluster algebras in Lecture II.

Q: Do all friezes arise from lightning bolts in this way?
A: No.

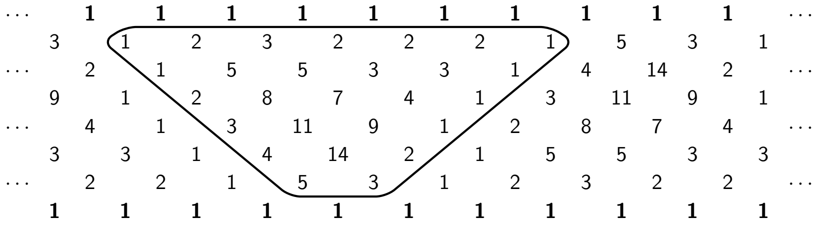
$$
\begin{array}{ccccccccccccccccccc}
 & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \\
\cdots & & & 1 & & 3 & & 1 & & 3 & & 1 & & 3 & & 1 & & & \cdots \\
 & & 2 & & 2 & & 2 & & 2 & & 2 & & 2 & & 2 & & 2 & & \\
\cdots & & & 3 & & 1 & & 3 & & 1 & & 3 & & 1 & & 3 & & & \cdots \\
 & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & &
\end{array}
$$

Periodicity

Phenomenon (Periodicity)

All frieze patterns are periodic under a glide reflection. In particular, each row is periodic with period (dividing) $n + 3$.

A fundamental domain for the glide reflection is as shown.



We will explain this phenomenon via cluster categories in Lecture III.

Quiddity sequences

Definition

The $(n+3)$-periodic sequence of integers in the first row of a frieze is called its quiddity sequence.

As with lightning bolts, a frieze is determined by its quiddity sequence using the diamond rule.

$$\begin{matrix} \cdots & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \cdots \\ & 3 & & 1 & & 2 & & 3 & & 2 & & 2 & & 2 & & 1 & & 5 & & 3 & & 1 & \end{matrix}$$

Thus we can start with any $(n+3)$-periodic sequence and try to construct a frieze from it, but there are many obstructions.

$$\begin{matrix} & b & \\ a & & d \\ & c & \end{matrix} \implies c = \frac{ad-1}{b}$$

This computation could give non-integer entries or 0. There is also no reason why the process should terminate with a trivial row of 1s at the expected time.

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& 3 & & 1 & & 2 & & 3 & & 2 & & 2 & & 2 & & 1 & & 5 & & 3 & & 1 & \\
& & {\color{blue}2} & & & & & & & & & & & & & & & & & & & &
\end{array}$$

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 & 3 & & 1 & & 2 & & 3 & & 2 & & 2 & & 2 & & 1 & & 5 & & 3 & & 1 & \\
 & & 2 & & \color{blue}{1} & & & & & & & & & & & & & & & & & &
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& 3 & & 1 & & 2 & & 3 & & 2 & & 2 & & 2 & & 1 & & 5 & & 3 & & 1 & \\
& & 2 & & 1 & & {\color{blue}5} & & & & & & & & & & & & & & & &
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\end{array}
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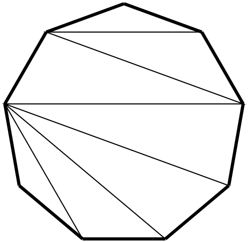
Triangulations

Consider a convex polygon with $n+3$ sides.

We choose a triangulation of this polygon—in other words, a maximal collection of pairwise non-crossing diagonals.



From this data, we can write an $(n+3)$-periodic sequence, by recording the number of triangles incident with each vertex of the polygon, taken in clockwise order.

$$\ldots, 1, 2, 3, 2, 2, 2, 1, 5, 3, \ldots$$

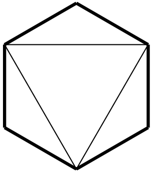
Classification

Theorem (Conway–Coxeter '71)

An $(n+3)$-periodic sequence is the quiddity sequence of a height n frieze if and only if it arises from a triangulation of an $(n+3)$-gon as on the preceding slide.

This gives a bijection between triangulations of polygons (up to rotation) and frieze patterns (up to horizontal translation).

If we break the rotational symmetry of the polygon (e.g. by numbering its vertices), we get a classification of friezes up to the glide symmetry from the periodicity phenomenon.



$\ldots, 1, 3, 1, 3, 1, 3, \ldots$

$$
\begin{array}{ccccccccccccccccc}
 & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & \\
\cdots & & 1 & & 3 & & 1 & & 3 & & 1 & & 3 & & 1 & & \cdots \\
 & 2 & & 2 & & 2 & & 2 & & 2 & & 2 & & 2 & & 2 & \\
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 & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} &
\end{array}
$$

An experiment

Instead of filling a lightning bolt with integers, we can use formal variables.

$$
\begin{array}{ccccccccccc}
 & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & \\
\cdots & & x_1 & & \frac{1+x_2}{x_1} & & \frac{1+x_1}{x_2} & & x_2 & & \cdots \\
 & \frac{1+x_1}{x_2} & & x_2 & & \frac{1+x_1+x_2}{x_1x_2} & & x_1 & & \frac{1+x_2}{x_1} & \\
\cdots & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \cdots
\end{array}
$$

All the entries surprisingly turn out to be Laurent polynomials.

$$\frac{1+\dfrac{1+x_1+x_2}{x_1x_2}}{\dfrac{1+x_2}{x_1}} = \frac{x_1(1+x_1+x_2+x_1x_2)}{x_1x_2(1+x_2)} = \frac{(1+x_1)(1+x_2)}{x_2(1+x_2)} = \frac{1+x_1}{x_2}$$

This Laurent phenomenon implies integrality, by setting each $x_i = 1$.

The Laurent polynomials appearing are cluster variables in a cluster algebra (of type A_n, with n the height of the frieze) – more on this next time.

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	1		**1**		**1**		**1**		**1**	
···		1		2		2		1		···
	2		1		3		1		2	
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From frieze patterns to cluster categories

Part II: Cluster algebras

Matthew Pressland

University of Leeds

LMS Autumn Algebra School
08.10.2020

Fomin and Zelevinsky

Сергей Фомин
Photo: Wikimedia Commons

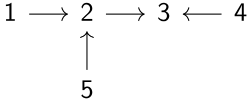
Андрей Зелевинский
Photo: Renate Schmid, © MFO

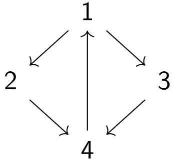
Quivers

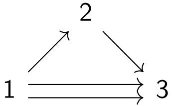
A cluster algebra is a commutative algebra (with extra combinatorial structure) defined starting from some combinatorial initial data.

For us, this combinatorial data will be a quiver, although more generality is possible.

A quiver is a directed graph—formally, it is a tuple $Q = (Q_0, Q_1, h, t)$, where $Q_0 = \{1, \ldots, n\}$ is the vertex set, Q_1 the arrow set, and $h, t \colon Q_1 \to Q_0$ specify the heads and tails of arrows.







Definition

A *cluster quiver* is a quiver Q without oriented cycles of length 1 or 2. In other words, no arrow $a \in Q_1$ can have $h(a) = t(a)$ (there are *no loops*) and the configuration $i \rightleftarrows j$ is not permitted (there are *no 2-cycles*).

Mutation

Definition

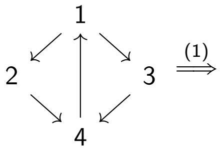
Let Q be a cluster quiver and pick $k \in Q_0$. The *mutation* $\mu_k Q$ of Q at k is obtained via the following procedure.

(1) For each length 2 path $i \longrightarrow k \longrightarrow j$, add an arrow $i \longrightarrow j$.

(2) Reverse the direction of all arrows incident with k.

(3) Choose a maximal set of 2-cycles, and remove all arrows appearing in them.

For example, we mutate the following quiver at vertex 1:



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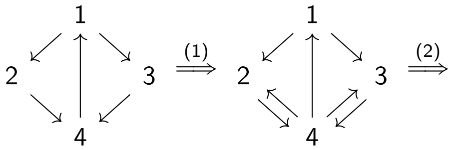

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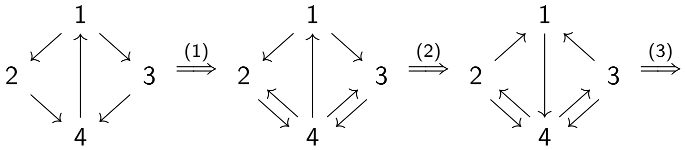

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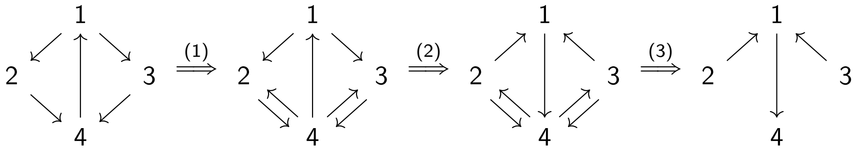

Mutating twice at the same vertex recovers the original quiver.

Cluster algebras

Let $\mathbb{Q}(x_1, \ldots, x_n)$ be the field of rational functions in x_i, $i \in \{1, \ldots, n\}$.

A *seed* is a cluster quiver Q with vertex set $Q_0 = \{1, \ldots, n\}$, together with a free generating set $\{f_1, \ldots, f_n\} \subseteq \mathbb{Q}(x_1, \ldots, x_n)$ indexed by Q_0.

Define $\mu_k(Q, \{f_i\}) = (\mu_k Q, \{f_i'\})$ where

$$f_i' = \begin{cases} f_i, & i \neq k, \\ \dfrac{1}{f_k}\Big(\displaystyle\prod_{k \to j} f_j + \prod_{\ell \to k} f_\ell\Big), & i = k. \end{cases}$$

Mutating twice at the same vertex recovers the original seed.

A cluster quiver Q with $Q_0 = \{1, \ldots, n\}$ has *initial seed* $s_0 = (Q, \{x_i\})$. Let $\mathcal{S}_Q$ be the set of all seeds obtained from s_0 by a finite sequence of mutations.

Definition

The *cluster algebra* A_Q of Q is the $\mathbb{Q}$-subalgebra of $\mathbb{Q}(x_1, \ldots, x_n)$ generated by all functions appearing in all seeds in $\mathcal{S}_Q$.

Cluster algebras

Definition

The *cluster algebra* A_Q of Q is the $\mathbb{Q}$-subalgebra of $\mathbb{Q}(x_1, \ldots, x_n)$ generated by all functions appearing in all seeds in $\mathcal{S}_Q$.

A_Q is a commutative algebra, with extra structure:

(1) A distinguished set of generators, the rational functions appearing in seeds in $\mathcal{S}_Q$: these are called *cluster variables*.

(2) A grouping of these generators into the (overlapping) n element sets $\{f_i\}$ in the seeds in $\mathcal{S}_Q$: these sets are called *clusters*.

While the definition is weird, many interesting rings are isomorphic to cluster algebras: coordinate rings of the Grassmannian, more general flag varieties, cells in decompositions of these, etc.

In that context $\mathbb{Q}$ is replaced by $\mathbb{C}$, and some vertices of Q are declared *frozen*.

Mutations are not performed at frozen vertices, and so the corresponding variables x_i appear in all clusters.

A_2 example

For $Q = 1 \longrightarrow 2$, the seeds of A_Q are

$$\left(1 \longrightarrow 2, \left\{x_1, x_2\right\}\right)$$

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A_2 example

For $Q = 1 \longrightarrow 2$, the seeds of $\mathcal{A}_Q$ are

$$\left(1 \longrightarrow 2, \left\{x_1, x_2\right\}\right)$$

$$\left(1 \longleftarrow 2, \left\{x_1, \frac{1+x_1}{x_2}\right\}\right) \longleftrightarrow \left(1 \longrightarrow 2, \left\{x_1, x_2\right\}\right) \longleftrightarrow \left(1 \longleftarrow 2, \left\{\frac{1+x_2}{x_1}, x_2\right\}\right)$$

$$\left(1 \longleftarrow 2, \left\{\frac{1+x_2}{x_1}, x_2\right\}\right) \longleftrightarrow \left(1 \longrightarrow 2, \left\{\frac{1+x_2}{x_1}, \frac{1+x_1+x_2}{x_1x_2}\right\}\right) \longleftrightarrow \left(1 \longleftarrow 2, \left\{\frac{1+x_1}{x_2}, \frac{1+x_1+x_2}{x_1x_2}\right\}\right)$$

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The five cluster variables are the Laurent polynomials that appeared in our experiment at the end of the last lecture, expressing frieze entries in terms of the entries in a lightning bolt.

Laurent phenomenon

Theorem (Fomin–Zelevinsky '02)

Let Q be a cluster quiver. Then every cluster variable in A_Q is a Laurent polynomial in any cluster, and in particular in the initial variables $\{x_1, \ldots, x_n\}$.

Proof.

A combinatorial proof is given in Fomin–Zelevinsky's original 2002 paper. In 2015, Gross–Hacking–Keel gave an alternative proof by realising cluster variables as regular functions on a somewhat complicated geometric object.

This means that if we evaluate the initial cluster variables x_i to 1, all cluster variables take integer values.

In a frieze of height n, we can write formulae for arbitrary entries in terms of the entries in a given lightning bolt.

If we can prove that these formulae are cluster variables in a cluster algebra, the Laurent phenomenon will imply integrality of friezes.

Finite type classification

Theorem (Fomin–Zelevinsky '03)

A cluster algebra A_Q has finitely many cluster variables (finite type) if and only if Q is related by a sequence of mutations to a quiver obtained by choosing an orientation of one of the following graphs (simply laced Dynkin diagrams).

A_n : $1 — 2 — \cdots — n-1 — n$

D_n : $1 — 2 — \cdots — n-1$, with n attached below 2

E_6 : $1 — 2 — 3 — 4 — 5$, with 6 attached below 3

E_7 : $1 — 2 — 3 — 4 — 5 — 6$, with 7 attached below 3

E_8 : $1 — 2 — 3 — 4 — 5 — 6 — 7$, with 8 attached below 3

Dynkin diagrams also classify finite root systems (among many other things).

The number of non-initial cluster variables in a finite type cluster algebra is the number of positive roots in the root system with matching Dynkin diagram.

Integrality for friezes

Theorem

Given a frieze of height n, the formulae expressing arbitrary entries in terms of those in a lightning bolt are given by cluster variables in A_Q for Q a quiver of type A_n.

Corollary

Starting from a lightning bolt with entries set to 1, we will always obtain a valid frieze pattern, i.e. the other entries will be positive integers. Moreover, the entries of a frieze pattern take only finitely many different values.

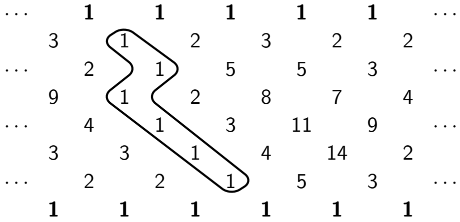
In the rest of the lecture, we give a sketch of the proof of theorem. The first step is to choose the right quiver.

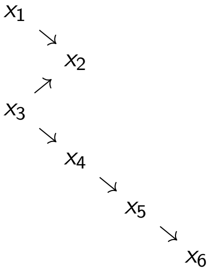
The quiver

Our quiver has vertices $1, \ldots, n$, with vertex i corresponding to the i-th non-trivial row of the frieze.

For each $i < n$, we draw an arrow $i \to i+1$ if the lightning bolt entry in row $i+1$ is to the right of that in row i, and $i+1 \to i$ otherwise.

This quiver Q has underlying graph A_n, and we consider the cluster algebra A_Q.





Since Q has no oriented cycles, it must have a source, say k.

This corresponds to three lightning bolt (or trivial) entries forming the left-hand part of a diamond.

Mutation

We mutate at the source: this amounts to reversing the incident arrows, creating a sink, and obtaining the new variable

$$x_k' = \frac{x_{k-1}x_{k+1} + 1}{x_k},$$

where if $k = 1$ or $k = n$ we interpret the undefined variable on the right-hand side as 1.

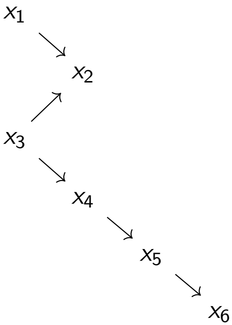
Key observation: $\begin{smallmatrix} & x_{k-1} & \\ x_k & & x_k' \\ & x_{k+1} & \end{smallmatrix}$ satisfies the SL_2 diamond rule.

This means that if we specialise each x_j to the entry in the j-th row of the lightning bolt, the cluster variable x_k' will be specialised to the entry directly to the right of the lightning bolt entry in row k.

Changing our lightning bolt by moving the entry in the k-th row to the right (which is legal since k is a source in the quiver), we get a new lightning bolt whose quiver is $\mu_k Q$.

Mutation

Continuing in this way, we see that all entries to the right of our lightning bolt are cluster variables in A_Q.



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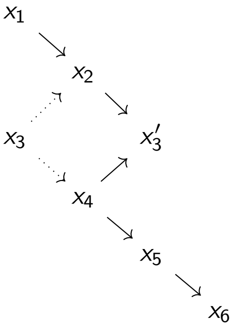

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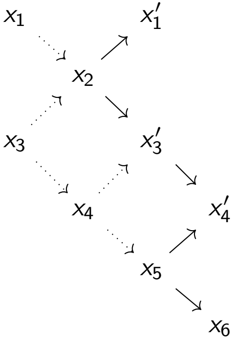

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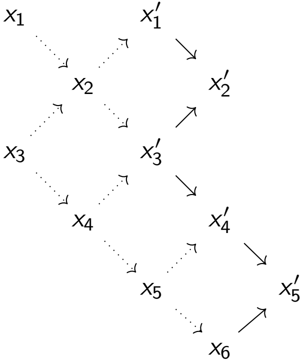
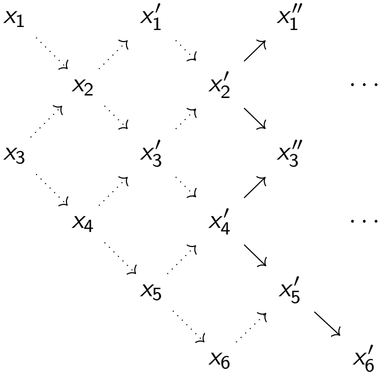


Mutation

Continuing in this way, we see that all entries to the right of our lightning bolt are cluster variables in A_Q.



Mutating at sinks instead of sources gives the argument for entries to the left.

From frieze patterns to cluster categories

Part III: Cluster categories

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University of Leeds

LMS Autumn Algebra School
09.10.2020

Buan, Marsh, Reineke, Reiten and Todorov (BMRRT)


Aslak Bakke Buan


Bethany Marsh


Markus Reineke


Idun Reiten
Photo: Renate Schmid, © MFO


Gordana Todorov

Introduction

Our aim in the last lecture is to describe a category which can be used to study cluster algebras.

To keep technicality to a minimum, we stick to the cluster category $\mathcal{C}_Q$ of an acyclic quiver Q, as introduced by Buan, Marsh, Reineke, Reiten and Todorov.

More general cluster categories, defined from a quiver with cycles (and some extra data) are defined by Amiot.

We will be most interested in the case that Q has underlying graph A_n, in which case $\mathcal{C}_Q$ was also constructed, in a slightly different way, by Caldero, Chapoton and Schiffler.

This construction will explain the second phenomenon from the first lecture, namely that frieze patterns are periodic under a glide reflection.

It will also give us a second classification of frieze patterns, explaining the origin of friezes not obtained by setting lightning bolt entries to 1.

Categories

A category $\mathcal{C}$ consists of

(1) a set of objects $\mathrm{Ob}(\mathcal{C})$ (we write $X \in \mathcal{C}$ for $X \in \mathrm{Ob}(\mathcal{C})$),

(2) a set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ of morphisms $X \to Y$ for any pair of objects $X, Y \in \mathcal{C}$, and

(3) an associative composition law consisting of maps
$\circ\colon \mathrm{Hom}_{\mathcal{C}}(Y, Z) \times \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$ for each triple $X, Y, Z \in \mathcal{C}$.

Each object X has an identity morphism $1_X \in \mathrm{Hom}_{\mathcal{C}}(X, X)$ such that $1_X \circ f = f$ and $g \circ 1_X = g$ whenever these compositions are defined.

Our categories will be $\mathbb{K}$-linear for a field $\mathbb{K}$, meaning that each $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is a $\mathbb{K}$-vector space, and the composition maps are $\mathbb{K}$-bilinear.

We fix an acyclic quiver Q for the rest of the lecture.

We will describe three associated categories, each constructed from the previous one: the category of representations of Q, the bounded derived category of Q, and finally the cluster category of Q.

Quiver representations

Definition

A *representation* (V, f) of Q consists of a finite-dimensional $\mathbb{K}$-vector space V_i for each $i \in Q_0$, and a $\mathbb{K}$-linear map $f_a \colon V_{t(a)} \to V_{h(a)}$ for each $a \in Q_1$.

A morphism $\varphi \colon (V, f) \to (W, g)$ of representations consists of linear maps $\varphi_i \colon V_i \to W_i$ for $i \in Q_0$ such that the diagram

$$\begin{array}{ccc} V_{t(a)} & \xrightarrow{f_a} & V_{h(a)} \\ {\scriptstyle \varphi_{t(a)}}\downarrow & & \downarrow{\scriptstyle \varphi_{h(a)}} \\ W_{t(a)} & \xrightarrow[g_a]{} & W_{h(a)} \end{array}$$

commutes for any $a \in Q_1$.

The direct sum $(V, f) \oplus (W, g)$ has vector spaces $(V \oplus W)_i = V_i \oplus W_i$ and linear maps $(f \oplus g)_a = \left(\begin{smallmatrix} f_a & 0 \\ 0 & g_a \end{smallmatrix}\right)$.

We say (V, f) is indecomposable if it is non-zero and (V, f) is not isomorphic to a direct sum of two non-zero representations.

The category of representations

The category $\operatorname{rep} Q$ has representations of Q as objects, with morphisms the morphisms of representations as defined on the previous slide.

This category is *abelian*: this includes the properties that

(1) its morphism spaces are abelian groups, and it has a well-defined direct sum operation on objects,

(2) morphisms have well-defined kernels and cokernels, and

(3) every injective morphism is a kernel, and every surjective morphism is a cokernel.

These properties mean that we can apply a general construction of Verdier to $\operatorname{rep} Q$: we can take its *bounded derived category*.

This construction can get complicated. Since we will only apply it to $\operatorname{rep} Q$, which is quite a simple category, we will cheat and give an ad hoc definition for this special case.

A fuller description of the construction in general can be found in the appendix of the lecture notes.

The bounded derived category

For each object $V \in \operatorname{rep} Q$ and integer $i \in \mathbb{Z}$, introduce the formal symbol $\Sigma^i V$.

The bounded derived category $\mathcal{D}^{\mathrm{b}}(Q)$ of Q has as objects formal direct sums of these symbols.

We define morphisms between the symbols using extension groups in $\operatorname{rep} Q$

$$\operatorname{Hom}_{\mathcal{D}^{\mathrm{b}}(Q)}(\Sigma^i V, \Sigma^j W) := \operatorname{Ext}_Q^{j-i}(V, W),$$

and extend to direct sums via the formulae

$$\operatorname{Hom}_{\mathcal{D}^{\mathrm{b}}(Q)}(X_1 \oplus X_2, Y) = \operatorname{Hom}_{\mathcal{D}^{\mathrm{b}}(Q)}(X_1, Y) \oplus \operatorname{Hom}_{\mathcal{D}^{\mathrm{b}}(Q)}(X_2, Y),$$
$$\operatorname{Hom}_{\mathcal{D}^{\mathrm{b}}(Q)}(X, Y_1 \oplus Y_2) = \operatorname{Hom}_{\mathcal{D}^{\mathrm{b}}(Q)}(X, Y_1) \oplus \operatorname{Hom}_{\mathcal{D}^{\mathrm{b}}(Q)}(X, Y_2).$$

Composition is given by cup product of extensions.

As our notation suggests, there is an autoequivalence Σ of $\mathcal{D}^{\mathrm{b}}(Q)$ which takes $\Sigma^i V$ to $\Sigma^{i+1} V$ and is the identity on morphisms.

$$\operatorname{Hom}_{\mathcal{D}^{\mathrm{b}}(Q)}(\Sigma^{i+1} V, \Sigma^{j+1} W) = \operatorname{Ext}_Q^{j-i}(V, W) = \operatorname{Hom}_{\mathcal{D}^{\mathrm{b}}(Q)}(\Sigma^i V, \Sigma^j W)$$

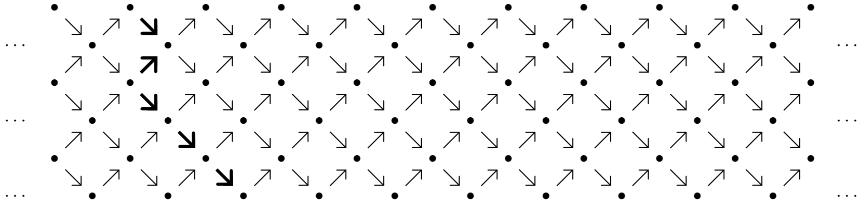
The repetition quiver

Now assume Q is a Dynkin quiver (i.e. its underlying graph is a Dynkin diagram). In this case we can describe $\mathcal{D}^{\mathrm{b}}(Q)$ more combinatorially.

Let $\mathbb{Z}Q$ be the *repetition quiver* of Q: its vertices are (i, n) for $i \in Q_0$ and $n \in \mathbb{Z}$, and its arrows are

$$a_n \colon (t(a), n) \to (h(a), n)$$
$$a_n^* \colon (h(a), n) \to (t(a), n+1)$$

for $a \in Q_1$ and $n \in \mathbb{Z}$.



When Q has type $\mathrm{A_n}$, it is natural to think of a height n frieze pattern (excluding trivial rows) as a function on the vertices of $\mathbb{Z}Q$.

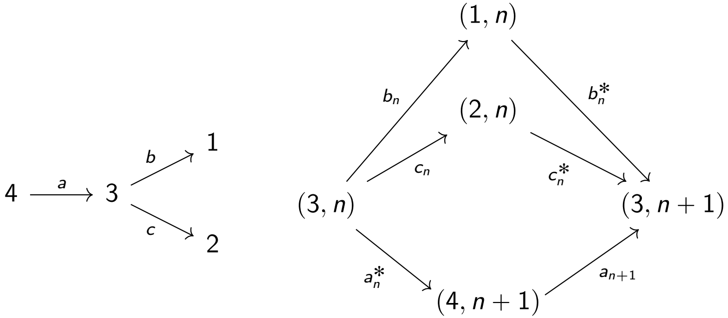
Note that $\mathbb{Z}Q$ is independent of the orientation of Q in this case.

Meshes

Each vertex (i, n) of $\mathbb{Z}Q$ gives rise to the *mesh relation*

$$\sum_{a:h(a)=i} a_{n+1}a_n^* - \sum_{b:t(a)=i} b_n^* b_n,$$

a formal linear combination of paths (which we read from right to left).

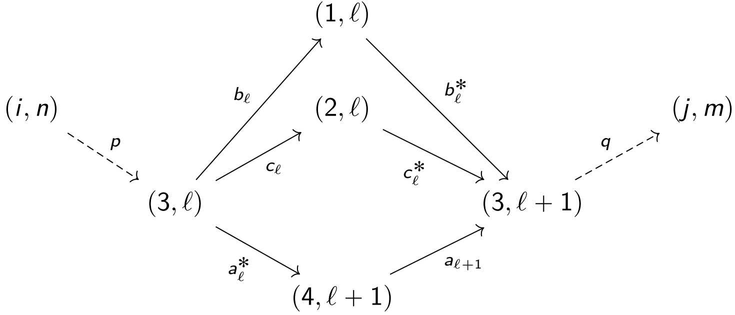


$$a_{n+1}a_n^* - b_n^* b_n - c_n^* c_n$$

The mesh category

The *mesh category* $\mathcal{D}_Q$ has as objects formal direct sums of vertices of the repetition quiver $\mathbb{Z}Q$.

Given vertices (i, n) and (j, m), the vector space $\mathrm{Hom}_{\mathcal{D}_Q}((i, n), (j, m))$ is spanned by paths $(i, n) \to (j, m)$ in $\mathbb{Z}Q$, subject to mesh relations.



$\implies q(a_{\ell+1} a_\ell^* - b_\ell^* b_\ell - c_\ell^* c_\ell)p = 0$ in $\mathrm{Hom}_{\mathcal{D}_Q}((i, n), (j, m))$.

Morphisms between direct sums are defined as in $\mathcal{D}^{\mathrm{b}}(Q)$.

Translation

The quiver $\mathbb{Z}Q$ has a symmetry τ, with

$$\tau\colon (i,n) \mapsto (i,n-1),$$
$$\tau\colon a_n \mapsto a_{n-1},$$
$$\tau\colon a_n^* \mapsto a_{n-1}^*.$$

This symmetry respects mesh relations, and so is an autoequivalence of $\mathcal{D}_Q$.

Theorem (Happel)

If Q is a Dynkin quiver, there is an equivalence of categories $\mathcal{D}^{\mathrm{b}}(Q) \xrightarrow{\sim} \mathcal{D}_Q$.

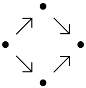
This means τ can be made into an autoequivalence of $\mathcal{D}^{\mathrm{b}}(Q)$.

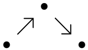
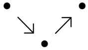
This auto-equivalence can also be defined intrinsically—$\mathcal{D}^{\mathrm{b}}(Q)$ has almost split sequences in the sense of Auslander–Reiten theory, and τ is the resulting Auslander–Reiten translation.

We will be most interested in quivers of type A_n, for which we can use the easier description $\mathcal{D}_Q$ of the bounded derived category.

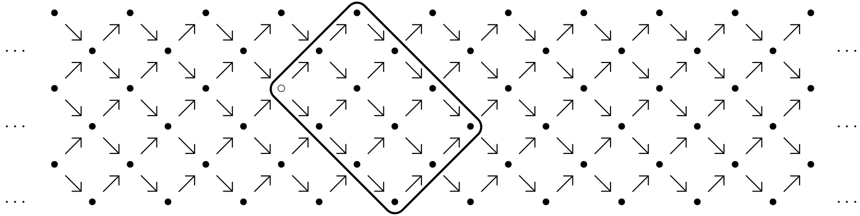
Computing morphisms

There are two kinds of mesh relation in type A_n.

First, each square  commutes or anti-commutes.

At the edges of the strip, compositions  and  are zero.

This allows us to combinatorially compute all morphism spaces between indecomposables.

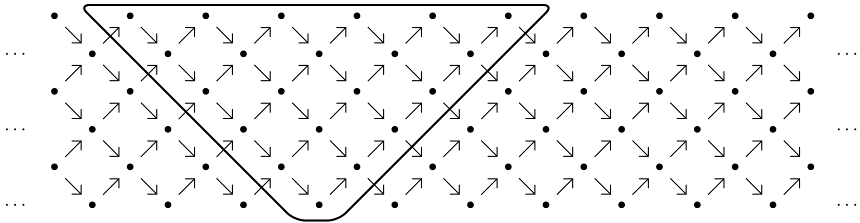


The object $\circ$ has a 1-dimensional space of morphisms to each object in the rectangle.

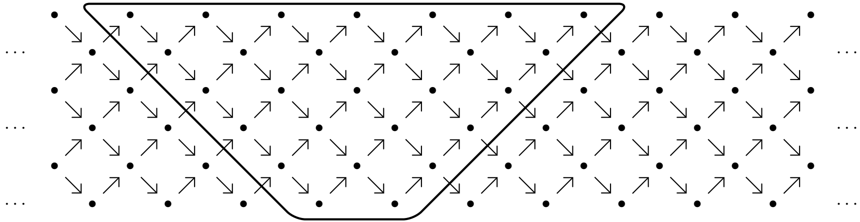
Symmetry

Since $\mathcal{D}^{\mathrm{b}}(Q) \xrightarrow{\sim} \mathcal{D}_Q$, we can make Σ into an autoequivalence of $\mathcal{D}_Q$.

In type A_n, the equivalence Σ acts on $\mathbb{Z}Q$ as a glide reflection to the right with fundamental domain as shown.



Thus $\Sigma^{-1} \circ \tau$ acts by a glide reflection with fundamental domain



Orbit category

Thinking of a frieze as a function on vertices of $\mathbb{Z}Q$ (or indecomposable objects of $\mathcal{D}_Q$, or of $\mathcal{D}^{\mathrm{b}}(Q)$), we want to see that it is invariant under $\Sigma^{-1} \circ \tau$.

This means it should be a function on indecomposable objects of the following orbit category.

Definition (BMRRT)

For an acyclic quiver Q, the cluster category $\mathcal{C}_Q$ is the orbit category

$$\mathcal{C}_Q := \mathcal{D}^{\mathrm{b}}(Q)/(\Sigma^{-1} \circ \tau).$$

This has the same objects as $\mathcal{D}^{\mathrm{b}}(Q)$, with morphisms

$$\mathrm{Hom}_{\mathcal{C}_Q}(X, Y) = \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{D}^{\mathrm{b}}(Q)}(X, (\Sigma^{-1} \circ \tau)^n Y).$$

While $\mathcal{C}_Q$ has the same objects as $\mathcal{D}^{\mathrm{b}}(Q)$ and more morphisms, there are fewer isomorphism classes, so it is ‘smaller’.

An isomorphism class in $\mathcal{C}_Q$ is a $(\Sigma^{-1} \circ \tau)$-orbit of isomorphism classes in $\mathcal{D}^{\mathrm{b}}(Q)$.

Categorification

The symmetries Σ and τ descend to autoequivalences of $\mathcal{C}_Q$, where they coincide.

Definition

We say objects $X, Y \in \mathcal{C}_Q$ are *compatible* if $\mathrm{Hom}_{\mathcal{C}_Q}(X, \Sigma Y) = 0$.

An object $X \in \mathcal{C}_Q$ is *rigid* if it is compatible with itself.

An object X is *cluster-tilting* if the set of objects compatible with X is $\operatorname{add} X$, the closure of $\{X\}$ under direct sums, direct summands and isomorphisms.

Theorem (BMRRT, Caldero–Keller)

There is a bijection between the indecomposable rigid objects of $\mathcal{C}_Q$ and the cluster variables of A_Q.

This bijection sends compatible pairs of indecomposable rigid objects to cluster variables appearing in the same cluster.

In particular, it induces a bijection between cluster-tilting objects of $\mathcal{C}_Q$ and clusters of A_Q.

Periodicity

Implicit in the proof of theorem is the fact that, in type A_n, objects in a mesh give cluster variables satisfying the SL_2 diamond rule.

$$\begin{array}{ccccc} & & B & & \\ & \nearrow & & \searrow & \\ A & & & & D \\ & \searrow & & \nearrow & \\ & & C & & \end{array} \implies \varphi_A\varphi_D - \varphi_B\varphi_C = 1$$

We take $\varphi_B = 1$ if B is missing in the mesh, and similarly for C.

We can read the previous theorem as giving an assignment of cluster variables to indecomposables of $\mathcal{D}^{\mathrm{b}}(Q)$, in such a way that values are constant on $(\Sigma^{-1} \circ \tau)$-orbits, and the values on a mesh satisfy the SL_2 diamond rule.

From the previous lecture, we know that any frieze is obtained by specialising these cluster variables, and conclude that friezes are periodic under the glide reflection $\Sigma^{-1} \circ \tau$.

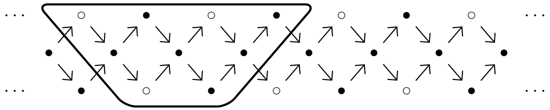
Classification

Theorem

Let Q be any A_n quiver. Then the friezes of height n are in bijection with the cluster-tilting objects of the cluster category $\mathcal{C}_Q$.

Indeed, given a cluster tilting object T, there is a unique frieze pattern taking the value 1 on each indecomposable summand of T.

Each lightning bolt gives a cluster-tilting object, but there are more.



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Final remarks

Since clusters can be mutated, so can cluster-tilting objects. This mutation can be defined intrinsically (BMRRT, Iyama–Yoshino) using that $\mathcal{C}_Q$ is a 2-Calabi–Yau triangulated category.

Comparing to the first classification of friezes, cluster-tilting objects in $\mathcal{C}_Q$, for Q of type A_n, are in bijection with triangulations of the $(n+3)$-gon.

This bijection can be made explicit, and the endomorphism algebra of a cluster-tilting object can be computed combinatorially from its triangulation (Caldero–Chapoton–Schiffler).

There is a generalisation to more general surfaces with boundary (Fomin–Shapiro–Thurston, Labardini-Fragoso).

Translating the mutation operation to triangulations, it becomes *flipping*.