Cluster categories from Postnikov diagrams

arXiv:1912.12475
and work in progress with İ. Çanakçı and A. King

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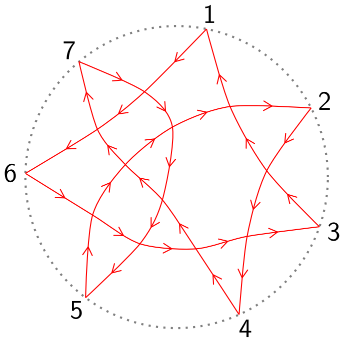
Postnikov diagrams

A Postnikov diagram D consists of n oriented strands in an oriented disc, connecting marked points $\{1, \ldots, n\}$ around the boundary, and satisfying



(P0) Each marked point is the source of one strand and the target of one strand.

(P1) The strands cross transversely, pairwise, and finitely many times.

(P2) Moving along each strand, the signs of its crossings with other strands alternate.

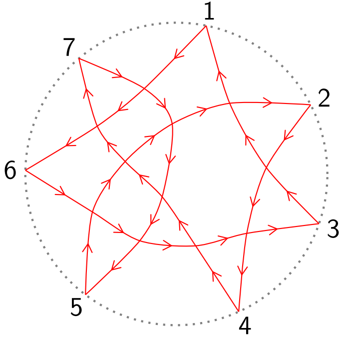
(P3) A strand does not cross itself.

(P4) If two strands cross twice, they are oriented in opposite directions between these crossings.

D determines $\sigma_D \in \mathfrak{S}_n$ by mapping the source of each strand to its target. In the example, $\sigma_D = (1, 6, 3)(2, 4, 7, 5)$.
Note: In this talk, we do not restrict to the permutations $i \mapsto i + k \mod n$.

The quiver

The strands of D cut the disc into regions, such that the orientation of strands around the boundary of each region is either *alternating*, *clockwise*, or *anticlockwise*.



D determines a quiver Q_D with

(Q_0) vertices corresponding to the alternating regions, and

(Q_1) arrows corresponding to crossings of strands.

Some vertices and arrows are on the boundary, and will sometimes play a different role to the others—we mark them in blue and call them *frozen*.

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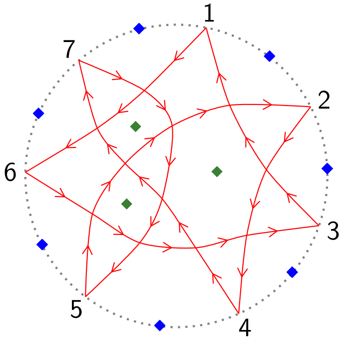

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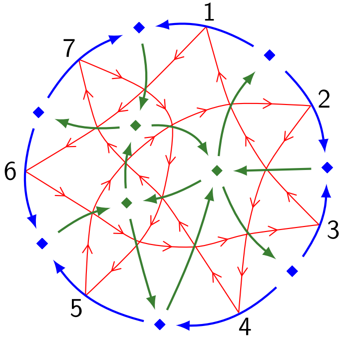

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A cluster algebra

The permutation σ_D is a Grassmann permutation, and hence determines a particular *positroid* subvariety $\Pi^\circ(\sigma_D) \subseteq \mathrm{Gr}_k^n$ of the Grassmannian of k-dimensional subspaces of $\mathbb{C}^n$ [Postnikov].

It aso determines a cluster algebra $\mathscr{A}_D$, with invertible frozen variables, via the quiver Q_D.

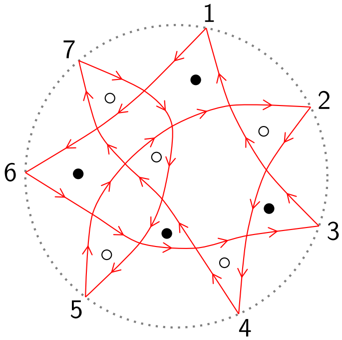
Theorem (Serhiyenko–Sherman-Bennett–Williams, Galashin–Lam)

There is an isomorphism $\mathscr{A}_D \xrightarrow{\sim} \mathbb{C}[\Pi^\circ(\sigma_D)]$, mapping the initial cluster variables to restrictions of Plücker coordinates.

For $\sigma_D \colon i \mapsto i + k \mod n$ (the *uniform permutation*), the variety $\Pi^\circ(\sigma_D)$ is dense in Gr_k^n, and the cluster algebra with non-invertible frozen variables attached to Q_D is isomorphic to the homogeneous coordinate ring $\mathbb{C}[\widehat{\mathrm{Gr}}_k^n]$. [Scott]

In this case, Jensen–King–Su have categorified the cluster algebra—our aim is to extend this to more general positroid varieties.

A non-commutative algebra



The oriented regions of D are either clockwise ($\circ$) or anticlockwise ($\bullet$).

Thus Q_D has a determined set of $\bullet$-cycles and $\circ$-cycles.

Let A_D be the $\mathbb{C}$-algebra determined by Q_D with relations as follows:

Each non-boundary (green) arrow a can be completed to either a $\bullet$-cycle or a $\circ$-cycle by unique paths $p_a^\bullet$ and p_a°; we impose the relation $p_a^\bullet = p_a^\circ$ for each a.

We call A_D the *dimer algebra* of D.

This is an example of a *frozen Jacobian algebra*, for the potential $W = \sum(\bullet\text{-cycles}) - \sum(\circ\text{-cycles})$.

Technical note: we take the complete path algebra of Q_D over $\mathbb{C}$, and the quotient by the closure of the ideal generated by the given relations.

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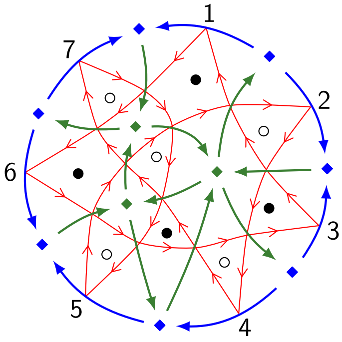

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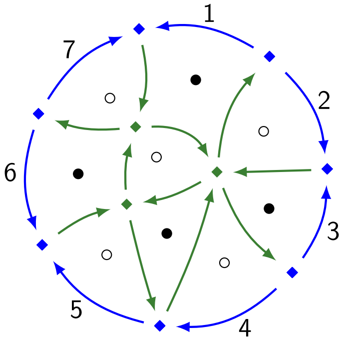

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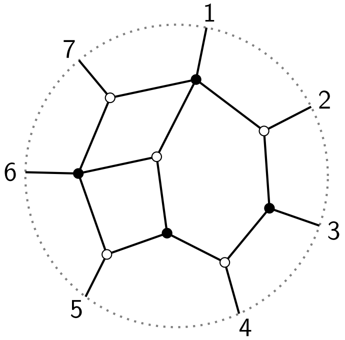
Interlude: dimer models

Consider a bipartite graph drawn in our disc, together with *half-edges* connecting some nodes to the boundary marked points.

This is called a dimer model, and it also determines a quiver and dimer algebra.

This construction makes sense on any oriented surface with or without boundary.



Theorem (Broomhead)

The dimer algebra of a consistent dimer model on the torus is bimodule 3-Calabi–Yau.

The dimer also determines strands—on the disc, consistency means that these strands are a Postnikov diagram. The dimer algebras constructed from the Postnikov diagram and the dimer model coincide.

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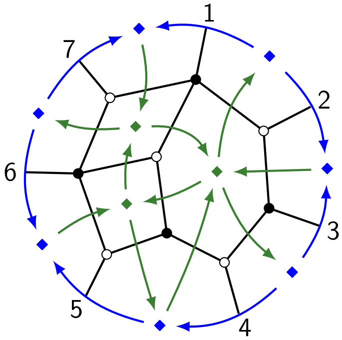

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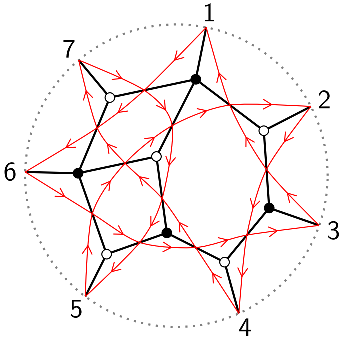

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The boundary algebra

Let $e = e^2 \in A_D$ be the sum of vertex idempotents at boundary vertices, and write $B_D = eA_De$ for the *boundary algebra*.

Theorem (Jensen–King–Su, Baur–King–Marsh)

When σ_D is the uniform permutation, the category

$$\mathrm{GP}(B_D) = \{X \in \operatorname{mod} B_D : \operatorname{Ext}^{>0}_{B_D}(X, B_D) = 0\}$$

categorifies the cluster algebra $\mathscr{A}_D$.

Note: this presentation is historically backwards. In practice, Jensen–King–Su proved the above theorem for an explicitly defined 'circle algebra' $C(k,n)$, depending only on the pair (k,n), which Baur–King–Marsh (slightly) later showed is isomorphic to B_D whenever σ_D is the uniform permutation.

With hindsight, we can try to repeat this trick, this time using the boundary algebra description of B_D as the (now more general) definition.

We can do this providing D is connected.

Categorification

Theorem

Let D be a connected Postnikov diagram in the disc, with dimer algebra $A = A_D$ and boundary algebra $B = B_D$. Then

(1) *B is Iwanaga–Gorenstein of Gorenstein dimension $\leqslant 3$; that is, B is Noetherian and $\operatorname{injdim}{}_B B, \operatorname{injdim} B_B \leqslant 3$. In particular $\mathrm{GP}(B)$ is a* ***Frobenius*** *category.*

(2) *The stable category $\underline{\mathrm{GP}}(B) = \mathrm{GP}(B)/\operatorname{proj} B$ is a* ***2-Calabi–Yau*** *triangulated category.*

(3) *$A = \operatorname{End}_B(eA)^{\mathrm{op}}$ and $eA \in \mathrm{GP}(B)$ is* ***cluster-tilting***, *that is*

$$\operatorname{add}(eA) = \{X \in \mathrm{GP}(B) : \operatorname{Ext}^1_B(X, eA) = 0\}.$$

This theorem follows from the following facts about the pair (A, e):

(1) A is Noetherian,

(2) A/AeA is finite-dimensional, and

(3) A is internally bimodule 3-Calabi–Yau with respect to e.

Internally Calabi–Yau algebras

The definition of A being internally bimodule 3-Calabi–Yau algebra is technical, and we omit it, but it implies that $\operatorname{gl.dim} A \leqslant 3$ and that

$$\operatorname{Ext}^i_A(X, Y) = \mathrm{D}\operatorname{Ext}^{3-i}_A(Y, X)$$

for $X, Y \in \operatorname{mod} A$ with $eY = 0$. ($\mathrm{D} = \operatorname{Hom}_{\mathbb{C}}(-, \mathbb{C})$)

The result for A_D is analogous to Broomhead's theorem for consistent dimer models on the torus.

The proof that A_D has this property uses that it is a frozen Jacobian algebra, and the following thinness property which uses connectedness of D.

Lemma (Çanakçı–King–P)

Let D be a connected Postnikov diagram. Then A_D has a central subalgebra $Z \cong \mathbb{C}[[t]]$, and for each pair of vertices i and j, there is an isomorphism $e_j A e_i \cong Z$ of Z-modules.

The required Noetherianity and finite-dimensionality also follow (more directly) from this lemma.

Boundary algebras

Since the cluster algebra $\mathscr{A}_D$, and the positroid variety $\Pi^\circ(\sigma_D)$, depend only on the permutation of σ_D, this should also be true of our category.

Proposition

If D and D' are connected Postnikov diagrams with $\sigma_D = \sigma_{D'}$, then $B_D \cong B_{D'}$, and so in particular $\mathrm{GP}(B_D) \simeq \mathrm{GP}(B_{D'})$.

This uses a result of Oh–Postnikov–Speyer; D and D' as in the Proposition are related by a sequence of local moves (which correspond to mutations of the quiver and in the cluster algebra!) which affect the isomorphism class of A_D, but not of the subalgebra $B_D = eA_De$.

The proof is really due to Baur–King–Marsh, who state the result for diagrams with $\sigma_D \colon i \mapsto i + k \mod n$.

The Jensen–King–Su category

We say a Postnikov diagram with n strands of 'average length' k has *type* (k,n). For example, if $\sigma_D \colon i \to i+k \mod n$ then D the strands have constant length k, so D has type (k,n).

Note: $\Pi^\circ(\sigma_D) \subseteq \mathrm{Gr}_k^n$ for (k,n) the type of D.

Proposition (Çanakçı–King–P)

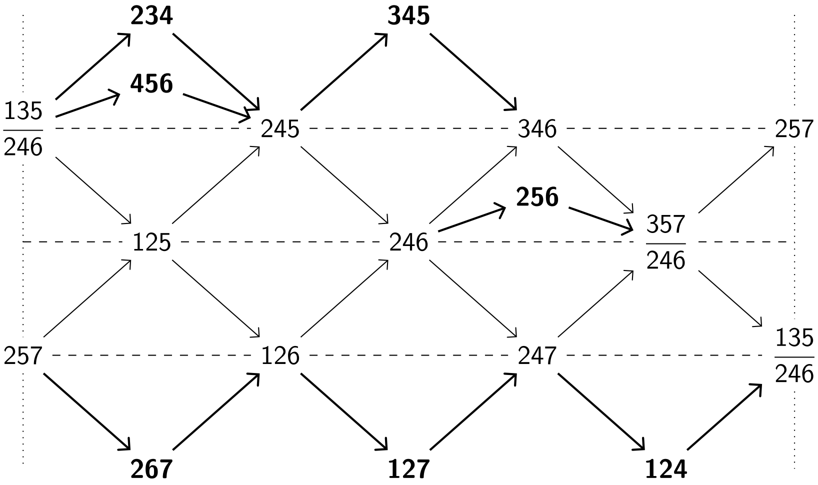
Let D be a diagram of type (k,n). Then there is a canonical ring morphism $C(k,n) \to B_D$, inducing a fully-faithful functor $\mathrm{GP}(B_D) \to \mathrm{GP}(C(k,n))$.

This means the categories we construct here all appear as full subcategories in Jensen–King–Su's Grassmannian cluster category, for the appropriate k and n.

To get the ring morphism: there is a canonical map $\Pi \to B_D$ for Π the preprojective algebra of type $\tilde{\mathrm{A}}_{n-1}$, since A_D is a frozen Jacobian algebra whose frozen subquiver is an orientation of this graph, and we check that this map factors over $C(k,n)$, which is by definition a quotient of Π.

Example

In the running example, $\mathrm{GP}(B_D)$ is as shown:



The names given to modules are as in Jensen–King–Su.

Advertising

17.09.20 (tomorrow), 2PM BST: FD-Seminar

Jenny August, on cluster categories for the infinite Grassmannian

21–25.09.20 and **05–09.10.20**: LMS Autumn Algebra School

Many interesting lecture series!

In the second week, I will give a more gentle introduction to cluster algebras and their categorification.

Thanks for listening!