Perfect matching modules for dimer algebras

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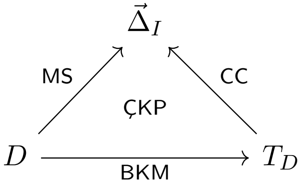
Setting

- Fix integers $1 \leq k < n$. We study the Grassmannian G_k^n of k-subspaces of $\mathbb{C}^n$, and the coordinate ring $\mathbb{C}[\hat{G}_k^n]$ of its affine cone.
- The 'standard' generators of $\mathbb{C}[\hat{G}_k^n]$ are Plücker coordinates Δ_I for $I \in \binom{n}{k} = \{I \subseteq \{1, \dots, n\} : |I| = k\}$.
- By work of Scott, $\mathbb{C}[\hat{G}_k^n]$ has a cluster algebra structure, in which all Δ_I are cluster variables.
- This cluster algebra is categorified by Jensen, King and Su, via certain bipartite graphs called dimer models.
- For each $I \in \binom{n}{k}$, there is a cluster monomial $\vec{\Delta}_I \in \mathbb{C}[\hat{G}_k^n]$; a *twisted Plücker coordinate*.
- This function can be computed from a dimer model D in two ways.

Two formulae



- MS = Marsh–Scott formula, which computes $\vec{\Delta}_I$ (as a Laurent polynomial in some Plücker coordinates determined by the dimer D) combinatorially from D.
- BKM = Baur–King–Marsh associate to D a maximal rigid object T_D in the JKS cluster category.
- Applying CC = the Caldero–Chapoton cluster character with respect to T_D produces the same Laurent polynomial expression for $\vec{\Delta}_I$.
- We, i.e. ÇKP, explain representation-theoretically why these two computations are really 'the same'.

Dimer models

- Take a disc with marked points $1, \ldots, n$ around its boundary.
- A dimer D is a bipartite graph in the interior of the disc, together with n 'half-edges' connecting black nodes to these marked points.
- $\#\{\text{black nodes}\} - \#\{\text{white nodes}\} = k$.
- D must be consistent: 'zig-zag paths form a Postnikov diagram'.
- A combinatorial rule (involving zig-zag paths) then attaches to each tile an element of $\binom{n}{k}$, and thus D determines a subset $\mathcal{C}(D) \subseteq \binom{n}{k}$.
- For our applications, we restrict to the case that the boundary tiles are labelled by the n cyclic intervals in $\binom{n}{k}$, in which case $\{\Delta_I : I \in \mathcal{C}(D)\}$ is a cluster of $\mathbb{C}[\hat{G}^n_k]$.

The Marsh–Scott formula

- A perfect matching μ of D is a set of edges of D (including half-edges) such that every node of D is incident with exactly one edge of μ.
- Since D has k more black vertices than white, any perfect matching must include exactly k half-edges, and the so the boundary marked points adjacent to these half-edges form a set $\partial\mu \in \binom{n}{k}$.
- The Marsh–Scott formula for $\vec{\Delta}_I$ is then

$$\vec{\Delta}_I = \sum_{\mu:\partial\mu=I} \Delta^{wt(\mu)}$$

for a vector $wt(\mu) \in \mathbb{Z}^{\mathcal{C}(D)}$ computed combinatorially from μ.

The JKS category

- D also determines an algebra A_D by taking the dual quiver, with relations $p_\alpha^+ = p_\alpha^-$ whenever there are paths p_α^+ and p_α^- completing an arrow α to a cycle around a black $(+)$ and white $(-)$ node.
- A_D is free of finite rank over a central subalgebra $Z \cong \mathbb{C}[[t]]$.
- Let e be the sum of vertex idempotents at the boundary tiles, and $B = eA_De$; this algebra is also free of finite rank over Z.

Theorem (Jensen–King–Su)

The category $\mathrm{CM}(B)$, of B-modules free of finite rank over Z, categorifies the cluster algebra $\mathbb{C}[\hat{G}_k^n]$. In particular, there is a bijection between isoclasses of rigid objects of $\mathrm{CM}(B)$ and cluster monomials of $\mathbb{C}[\hat{G}_k^n]$.

Theorem (Baur–King–Marsh)

The algebra B is independent of D, up to isomorphism. The B-module $T_D := eA_D$ is a maximal rigid object in $\mathrm{CM}(B)$, and $\mathrm{End}_B(T_D)^{\mathrm{op}} \cong A_D$.

The CC formula

- Fix a dimer model D, with corresponding maximal rigid object $T_D \in \mathrm{CM}(B)$, and set of Plücker labels $\mathcal{C}(D)$.
- Let $F = \mathrm{Hom}_B(T_D, -)$ and $G = \mathrm{Ext}^1_B(T_D, -)$; both are functors $\mathrm{CM}(B) \to \operatorname{mod} A_D$.
- Then the Caldero–Chapoton map (which gives the JKS bijection) is

$$\mathrm{CC}(X) = \sum_{N \leq GX} \Delta^{wt(N)} \qquad \left(\text{cf. MS: } \vec{\Delta}_I = \sum_{\mu : \partial\mu = I} \Delta^{wt(\mu)}\right)$$

- Here $wt(N) \in \mathbb{Z}^{\mathcal{C}(D)}$ is computed from projective resolutions of the A_D-modules FX and N.

MS=CC

- Let $I \in \binom{n}{k}$ and let $M_I \in \mathrm{CM}(B)$ be the object corresponding to the Plücker coordinate Δ_I; these modules are explicitly described by JKS.
- Each M_I has a 'canonical' projective cover P_I, yielding an exact sequence

$$0 \longrightarrow \Omega M_I \longrightarrow P_I \longrightarrow M_I \longrightarrow 0,$$

Proposition (Geiß–Leclerc–Schröer, Çanakçı–King–P)

$\mathrm{CC}(\Omega M_I) = \vec{\Delta}_I$.

- Hence, using the two formulae, we have

$$\sum_{\mu:\partial\mu=I} \Delta^{wt(\mu)} = \vec{\Delta}_I = \sum_{N\leq G\Omega M_I} \Delta^{wt(N)}$$

- Aim: use equality of the outer terms to deduce representation-theoretic information about A_D and B.

Perfect matching modules

$$\sum_{\mu:\partial\mu=I} \Delta^{wt(\mu)} = \sum_{N \leq G\Omega M_I} \Delta^{wt(N)}$$

- Let μ be a perfect matching of D.
- Define an A_D-module $\hat{N}_\mu$ by placing a copy of $Z = \mathbb{C}[[t]]$ at each vertex, and having arrows act by multiplication with t if they are dual to edges in μ, and by the identity otherwise.
- Applying F to the exact sequence defining ΩM_I gives an exact sequence

$$FP_I \xrightarrow{f} FM_I \xrightarrow{g} G\Omega M_I \longrightarrow 0$$

Theorem (Çanakçı–King–P, 'MS=CC')

The submodules of FM_I containing $\operatorname{im} f$ are precisely the $\hat{N}_\mu$ with $\partial\mu = I$. Thus, setting $N_\mu := g\hat{N}_\mu$, the assignment $\mu \mapsto N_\mu$ is a bijection $\{\mu : \partial\mu = I\} \xrightarrow{\sim} \{N \leq G\Omega M_I\}$. Moreover, $wt(\mu) = wt(N_\mu)$.