Internally Calabi–Yau Algebras

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Main Definition

- Let A be a (not necessarily finite dimensional) Noetherian $\mathbb{K}$-algebra, and let e be an idempotent of A.
- Throughout, we will write $\underline{A} = A/AeA$ (the interior algebra) and $B = eAe$ (the boundary algebra).

Definition

The algebra A is internally d-Calabi–Yau with respect to e if

(i) $\operatorname{gl.dim} A \leq d$, and

(ii) for any finite dimensional $M \in \operatorname{mod} \underline{A}$, and any $N \in \operatorname{mod} A$, there is a duality

$$\mathrm{D}\operatorname{Ext}^i_A(M,N) = \operatorname{Ext}^{d-i}_A(N,M)$$

for all i, functorial in M and N.

- Also a stronger definition of 'bimodule internally d-Calabi–Yau' involving complexes of A-modules (which we will see later, if there is time.)

Voidology

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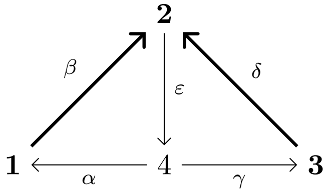
for all i, functorial in M and N.

- Setting $e=0$ recovers the (naïve) definition of a d-Calabi–Yau algebra.
- Setting $e=1$, (ii) becomes vacuous.
- If $e \neq 1$, (ii) $\implies \operatorname{gl.dim} A \geq d$, and so $\operatorname{gl.dim} A = d$ in this case.

Example ($d = 3$)



$$\varepsilon\beta = 0 = \varepsilon\delta$$

$$\beta\alpha = \delta\gamma$$

$$e = e_1 + e_2 + e_3$$

- $\underline{A} = \mathbb{K}$.
- $B = eAe$ is a quotient of the preprojective algebra of type A_3.

Origins

- Let $\mathcal{E}$ be a Frobenius category: an exact category with enough projectives and enough injectives, and such that projective and injective objects coincide.
- Then $\underline{\mathcal{E}} = \mathcal{E}/\operatorname{proj}\mathcal{E}$ is triangulated (Happel).
- Assume that $\mathcal{E}$ is idempotent complete, and $\underline{\mathcal{E}}$ is d-Calabi–Yau.
- Let $T \in \mathcal{E}$ be d-cluster-tilting, i.e.

$$\operatorname{add} T = \{X \in \mathcal{E} : \operatorname{Ext}^i_{\mathcal{E}}(X, T) = 0,\ 0 < i < d\}.$$

Theorem (Keller–Reiten)

If $\operatorname{gl.dim} \operatorname{End}_{\mathcal{E}}(T)^{\mathrm{op}} \leq d+1$, then it is internally $(d+1)$-Calabi–Yau with respect to projection onto a maximal projective summand.

Main Theorem

Theorem

Let A be a Noetherian algebra, and e an idempotent such that $\underline{A}$ is finite dimensional. Recall $B = eAe$. If A and A^{op} are internally $(d+1)$-Calabi–Yau with respect to e, then

(i) *B is Iwanaga–Gorenstein of Gorenstein dimension at most $d+1$, and so*

$$\mathrm{GP}(B) = \{X \in \operatorname{mod} B : \operatorname{Ext}^i_B(X, B) = 0,\ i > 0\}$$

is Frobenius,

(ii) *$eA \in \mathrm{GP}(B)$ is d-cluster-tilting, and*

(iii) *there are natural isomorphisms $A \cong \operatorname{End}_B(eA)^{\mathrm{op}}$ and $\underline{A} \cong \operatorname{End}_{\underline{\mathrm{GP}}(B)}(eA)^{\mathrm{op}}$.*

If A is bimodule internally $(d+1)$-Calabi–Yau with respect to e, then additonally

(iv) *$\underline{\mathrm{GP}}(B)$ is d-Calabi–Yau.*

Frozen Jacobian algebras

- Let Q be a quiver, and F a (not necessarily full) subquiver, called frozen.
- Let W be a linear combination of cycles of Q.
- For a cyclic path $\alpha_n \cdots \alpha_1$ of Q, define

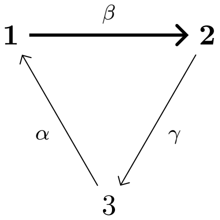
$$\partial_\alpha(\alpha_n \cdots \alpha_1) = \sum_{\alpha_i=\alpha} \alpha_{i-1} \cdots \alpha_1 \alpha_n \cdots \alpha_{i+1}$$

  and extend by linearity.
- The frozen Jacobian algebra $J(Q, F, W)$ is

$$J(Q, F, W) = \mathbb{C}Q / \langle \partial_\alpha W : \alpha \in Q_1 \setminus F_1 \rangle,$$

  where $\mathbb{C}Q$ denotes the complete path algebra of Q over $\mathbb{C}$.
- The frozen idempotent is $e = \sum_{i \in F_0} e_i$.

Example



F is the full subquiver on vertices 1 and 2.

$$W = \gamma\beta\alpha$$

$$e = e_1 + e_2$$

A bimodule resolution?

- Let A be a frozen Jacobian algebra, let $S = A/\mathfrak{m}(A)$ be the semisimple part of A, and write $\otimes = \otimes_S$. Write $\overline{Q}_i^{\mathrm{m}}$ for the dual S-bimodule to $Q_i \setminus F_i$.
- There is a natural complex

$$0 \to A\otimes\overline{Q}_0^{\mathrm{m}}\otimes A \to A\otimes\overline{Q}_1^{\mathrm{m}}\otimes A \to A\otimes Q_1\otimes A \to A\otimes Q_0\otimes A \to A \to 0$$

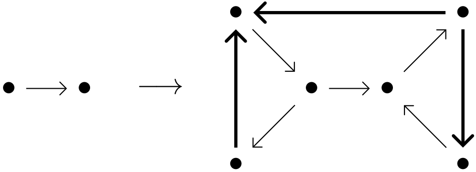
of A-bimodules (cf. Ginzburg and Broomhead for the case $F = \varnothing$).

Theorem

If this complex is exact, then A is bimodule internally 3-Calabi–Yau with respect to the frozen idempotent e.

A (double) principal coefficient construction

- Let $(\underline{Q}, \underline{W})$ be a Jacobi-finite quiver with potential.
- Construct (Q, F, W) by gluing triangles to vertices of $\underline{Q}$, rectangles along arrows of $\underline{Q}$:

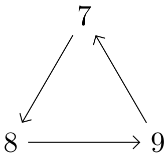


- $W = \underline{W} +$ triangles $-$ rectangles.

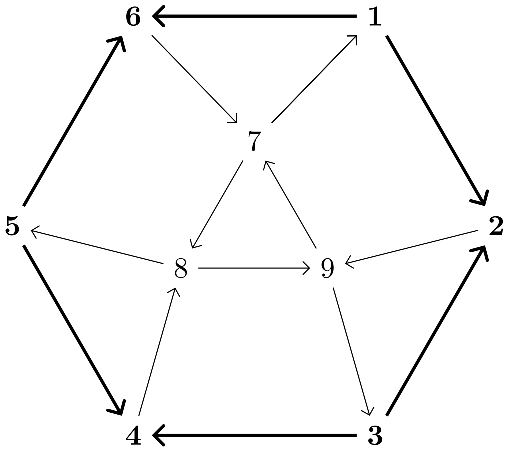
Theorem

Assume $J(\underline{Q}, \underline{W})$ can be graded with arrows in positive degree. Then $J(Q, F, W)$ is bimodule internally 3-Calabi–Yau with respect to e.

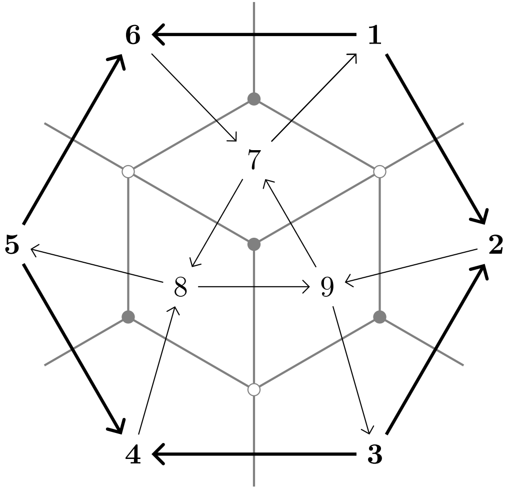
Example



Example



Example



- cf. Jensen–King–Su, Baur–King–Marsh, $(2,6)$-Grassmannian.

Bimodule version

- Write $A^{\varepsilon} = A \otimes_{\mathbb{K}} A^{\mathrm{op}}$, and $\Omega_A = \mathrm{RHom}_{A^{\varepsilon}}(A, A^{\varepsilon})$. Let $\mathcal{D}_{\underline{A}}(A)$ be the full subcategory of the derived category of A consisting of objects whose total cohomology is a finite-dimensional $\underline{A}$-module.

Definition

The algebra A is bimodule internally d-Calabi–Yau with respect to e if

(i) $\mathrm{p.dim}_{A^{\varepsilon}} A \leq d$, and

(ii) there is a triangle

$$A \to \Omega_A[d] \to C \to A[1]$$

in $\mathcal{D}(A^{\varepsilon})$, such that $\mathrm{RHom}_A(C, M) = 0 = \mathrm{RHom}_{A^{\mathrm{op}}}(C, N)$ for all $M \in \mathcal{D}_{\underline{A}}(A)$ and $N \in \mathcal{D}_{\underline{A}^{\mathrm{op}}}(A^{\mathrm{op}})$.

- If we can take $C = 0$, then $A \cong \Omega_A[d]$ is bimodule d-Calabi–Yau.

Consequences

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in $\mathcal{D}(A^\varepsilon)$, such that $\mathrm{RHom}_A(C, M) = 0 = \mathrm{RHom}_{A^{\mathrm{op}}}(C, N)$ for all $M \in \mathcal{D}_{\underline{A}}(A)$ and $N \in \mathcal{D}_{\underline{A}}(A^{\mathrm{op}})$.

- A is bimodule internally d-Calabi–Yau with respect to e if and only if the same is true for A^{op}.
- If A is bimodule internally d-Calabi–Yau with respect to e then

$$\mathrm{D}\,\mathrm{Hom}_{\mathcal{D}(A)}(M, N) = \mathrm{Hom}_{\mathcal{D}(A)}(N, M[d])$$

for any $N \in \mathcal{D}(A)$ and any $M \in \mathcal{D}_{\underline{A}}(A)$.
- In particular, such an A is internally d-Calabi–Yau with respect to e.