Cluster Automorphisms and Homogeneous Spaces

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Labelled Seeds

- Let $I = \{1, \ldots, n\}$.
- Let $\mathbb{F} \cong \mathbb{Q}(x_1, \ldots, x_n)$ be a purely transcendental field extension of $\mathbb{Q}$.

Definition

A *labelled seed* is a pair (Q, β) consisting of

- a quiver Q with vertex set I and without loops or 2-cycles, and
- a free generating set $\beta = (\beta_1, \ldots, \beta_n)$ (labelled by I) for $\mathbb{F}$ over $\mathbb{Q}$.

- The constant vertex set allows us to talk about *equality* of quivers; two quivers are equal if they have the same adjacency matrix.
- For example, $1 \to 2$ and $2 \to 1$ are *not* equal.
- The data of the n-tuple β is equivalent to the choice of an isomorphism $\beta\colon \mathbb{Q}(x_1, \ldots, x_n) \xrightarrow{\sim} \mathbb{F}$, via $\beta(x_i) = \beta_i$.

Mutations and Permutations

- Given $i \in I$, let $Q\mu_i$ be the Fomin–Zelevinsky mutation of Q at i. Let $\beta\mu_i$ be defined by

$$(\beta\mu_i)_j = \begin{cases} \beta_j, & j \neq i \\ \frac{\prod_{k\to i}\beta_k + \prod_{i\to k}\beta_k}{\beta_i}, & j = i \end{cases}$$

- Given $\sigma \in \mathfrak{S}_n$, let $(Q\sigma)_{ij} = Q_{\sigma(i)\sigma(j)}$ and $(\beta\sigma)_i = \beta_{\sigma(i)}$. (Here Q_{ij} is the number of arrows $i \to j$).
- Thus we obtain a right action of the *mutation group*

$$M_n = \mathfrak{S}_n \ltimes \langle \mu_1, \dots, \mu_n : \mu_i^2 = 1 \rangle$$

  (where $\sigma\mu_i = \mu_{\sigma(i)}\sigma$) on the set of all labelled seeds.
- The orbits are called *mutation classes*, and are homogeneous spaces for M_n.

Clusters

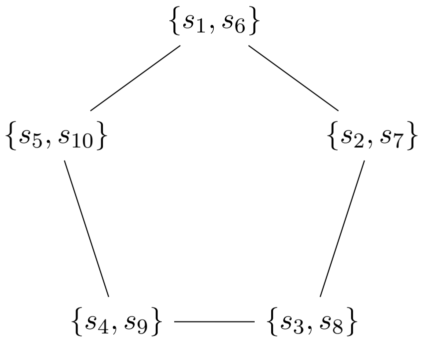
- Let $\mathcal{S}$ be a mutation class of labelled seeds; we abuse notation and write $Q \in \mathcal{S}$ and $\beta \in \mathcal{S}$ when $(Q, \beta) \in \mathcal{S}$.
- Define
$$\mathcal{X}(\mathcal{S}) = \bigcup_{\beta \in \mathcal{S}} \{\beta_1, \ldots, \beta_n\}.$$
- The *cluster algebra* $\mathcal{A}(\mathcal{S}) \subseteq \mathbb{F}$ is the subalgebra of $\mathbb{F}$ generated by $\mathcal{X}(\mathcal{S})$, and the set $\mathcal{X}(\mathcal{S})$ is the set of *cluster variables* of $\mathcal{A}(\mathcal{S})$.

Compare and Contrast

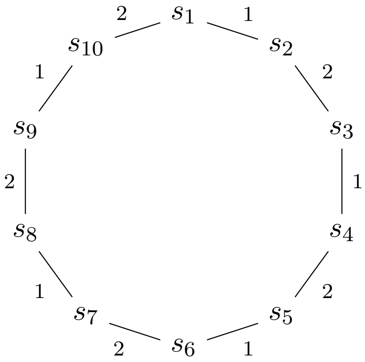
- Let $\mathbb{F} = \mathbb{Q}(x, y)$, and $s_1 = (Q, \beta)$ be the labelled seed with $Q = 1 \to 2$, $\beta_1 = x$ and $\beta_2 = y$.
- The seed $(Q', \gamma) = s_1 \cdot (1\,2) = s_1 \cdot \mu_1\mu_2\mu_1\mu_2\mu_1$ has quiver $Q' = 1 \leftarrow 2$, $\gamma_1 = y$ and $\gamma_2 = x$.
- We never identify this with s_1, but keep both labelled seeds and remember that they are related by the action of $\mathfrak{S}_2$.
- In total there are ten labelled seeds in the mutation class of s_1.

Compare and Contrast

In the unlabelled case, we obtain the familiar pentagon.



By labelling, we get a less familiar, but 'more homogeneous', decagon.



Regular Equivalence Relations

- Let X be a homogeneous space for a group G acting on the right, and let $\sim$ be an equivalence relation on X.

Definition

The equivalence relation $\sim$ is *regular* if whenever $x \sim y$, we have $x \cdot g \sim y \cdot g$ for all $g \in G$, and $\mathrm{Stab}_G(x) = \mathrm{Stab}_G(y)$.

Example

- The coarsest regular equivalence relation has $x \sim y \iff \mathrm{Stab}_G(x) = \mathrm{Stab}_G(y)$.
- Let $\mathcal{S}$ be a mutation class of labelled seeds, and say $(Q_1, \beta_1) \simeq (Q_2, \beta_2) \iff Q_1 = Q_2$. This relation is regular.
- For $\mathcal{S}$ as above, say $(Q_1, \beta_1) \approx (Q_2, \beta_2)$ if and only if Q_1 is obtained from Q_2 by reversing the orientation of some connected components (we say the quivers and seeds are *similar* in this case). This relation is regular.

G-Automorphisms

- The group $\mathrm{Aut}_G(X)$ of automorphisms of X commuting with the G-action acts on the left of X.

Proposition

A relation $\sim$ is regular if and only if there exists $W \leq \mathrm{Aut}_G(X)$ such that the $\sim$-classes are the W-orbits.

Example

The equivalence classes of the relation $x \sim y \iff \mathrm{Stab}_G(x) = \mathrm{Stab}_G(y)$ are the $\mathrm{Aut}_G(X)$-orbits.

Cluster Automorphisms

After Assem–Schiffler–Shramchenko

- For the rest of the talk, fix a mutation class $\mathcal{S}$, and write $\mathcal{A} = \mathcal{A}(\mathcal{S})$.

Definition/Proposition (Assem–Schiffler–Shramchenko)

An automorphism $f\colon \mathcal{A} \to \mathcal{A}$ is a *cluster automorphism* if for some (equiv. every) $(Q, \beta) \in \mathcal{S}$, there exists $(Q', \gamma) \in \mathcal{S}$ such that $f(\beta_i) = \gamma_i$ for all i, and $Q \approx Q'$. If $Q = Q'$, the cluster automorphism f is *direct*.

- So we obtain a group $\operatorname{Aut} \mathcal{A}$ of cluster automorphisms of $\mathcal{A}$, and a subgroup $\operatorname{Aut}^+(\mathcal{A})$ of direct cluster automorphisms.

Cluster Automorphisms

- Let $W \leq \mathrm{Aut}_{M_n}(\mathcal{S})$ be the group corresponding to the relation $\approx$ (similarity of quivers) and $W^+ \leq W$ the group corresponding to the relation $\simeq$ (equality of quivers).
- For $w \in W$, define $\alpha(w)\colon \mathbb{F} \to \mathbb{F}$ to be the unique automorphism extending $\alpha(w)(\beta_i) = \gamma_i$, where $w \cdot (Q, \beta) = (Q', \gamma)$.

Theorem (King–P)

The restriction of $\alpha(w)$ to $\mathcal{A}$ is a cluster automorphism, direct if and only if $w \in W^+$. Moreover α induces an isomorphism $W \xrightarrow{\sim} \mathrm{Aut}\,\mathcal{A}$, restricting to an isomorphism $W^+ \xrightarrow{\sim} \mathrm{Aut}^+\,\mathcal{A}$.

- Each subgroup $H \leq \mathrm{Aut}_{M_n}(\mathcal{S})$ also defines a groupoid $H\backslash\mathcal{S}$ with point groups isomorphic to H. The groupoid $W^+\backslash\mathcal{S}$ is Fock–Goncharov's cluster modular groupoid.

Small Mutation Classes

Definition

A mutation class $\mathcal{S}$ is *small* if it contains only finitely many quivers.

Theorem (King–P)

If $\mathcal{S}$ is small, then two seeds of $\mathcal{S}$ have similar quivers if and only if they have the same M_n-stabilizer, so $\mathrm{Aut}_{M_n}(\mathcal{S}) = W \cong \mathrm{Aut}\,\mathcal{A}$.

- If $\mathcal{S}$ contains a Dynkin type quiver (so $\mathcal{S}$ is finite) then the isomorphism class of $\mathrm{Aut}_{M_n}(\mathcal{S}) \cong \mathrm{Aut}\,\mathcal{A}$ has been computed by Assem–Schiffler–Shramchenko.
- It would be interesting to know if the property $W = \mathrm{Aut}_{M_n}(\mathcal{S})$ is equivalent to $\mathcal{S}$ being small.