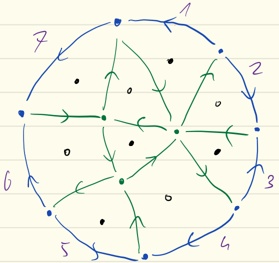
arXiv:2307.13369 Slides: bit.ly/PresslandFD

Positroid varieties via rep. theory

1) Positroids etc

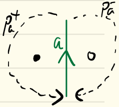
(Çanakçı-King-P remix, see also Postnikov, Oh-Postnikov-Speyer, Knutson-Lam-Speyer, ...)



Input: consistent dimer quiver Q on the disc, with faces $Q_2^{\bullet} \sqcup Q_2^{\circ}$.

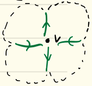
Dimer algebra $A = A_Q$:

$$A = \widehat{\mathbb{C}Q} / \overline{\langle p_a^+ = p_a^- : a \in Q_1 \text{ internal} \rangle}$$



$$= J(Q, F, W), \quad F = \partial Q, \quad W = \sum_{f \in Q_2^+} \partial f - \sum_{f \in Q_2^-} \partial f$$

For $v \in Q_0$, choose path $t_v : v \to v$ bounding a face
In A, indep. of choice, get $t = \sum_{v \in Q_0} t_v$.



$\Rightarrow A$ is a Z-algebra for $Z = \mathbb{C}[\![t]\!]$.

Λ Z-algebra $\rightsquigarrow \mathrm{CM}(\Lambda) = \{ M \in \operatorname{mod} \Lambda : {}_Z M \text{ free} + \text{f.g.} \}$
(i.e. ${}_Z M \in \mathrm{CM}(Z)$)

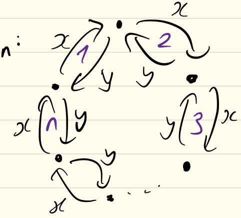
For $M \in \mathrm{CM}(A)$, $\mathrm{rk}_Z M(v) =: \mathrm{rk} M$ is indep. of $v \in Q_0$.

Prop (ÇKP) Consistency $\Rightarrow P_v = Ae_v \in \mathrm{CM}(A)$, $\mathrm{rk} P_v = 1$ $\forall_{v \in Q_0}$
($\Leftarrow$ [Berggren-Serhiyenko])

In particular, $A \in \mathrm{CM}(A)$.

$A \ni e = \sum_{v \in F_0 = \partial Q_0} e_v \rightsquigarrow B := eAe$ boundary algebra

Q_n:



$\Pi_n = \widehat{\mathbb{C}Q_n} / \overline{(xy - yx)}$
($\hat{A}$-type preprojective algebra)

For $0 < k < n$:
$C = C_{k,n} = \Pi_n / \overline{(y^k - x^{n-k})}$
Z-algebra for $t = xy$.

$\binom{n}{k} := \{ I \subseteq \{1,\dots,n\} : |I| = k \}$.

Given $I \in \binom{n}{k}$, define $M_I \in CM(C)$ by

$$M_I(v) = Z \ \forall v, \qquad M_I(x_i) = \begin{cases} t, & i \in I \\ 1, & i \notin I \end{cases} \qquad M_I(y_i) = \begin{cases} 1, & i \in I \\ t, & i \notin I \end{cases}$$

Thm (Jensen-King-Su '16)

1) $C \in CM(C)$ (for Z-algebra structure $t = xy = yx$).
2) $M \in CM(C)$ rk 1 $\iff M \cong M_I$ for $I \in \binom{n}{k}$.

(Note: $M_I \cong M_J \Rightarrow I = J$.)

Rem Similar classification of $N \in CM(A)$ rk 1 in terms of perfect matchings of Q (ÇKP) ⇝ critical ingredient in proofs!

Obs ∃ canonical map $\Pi_n \to B$.

Prop (ÇKP) $\exists!\, 0 < k < n$ s.t.

$$\begin{tikzcd} \Pi_n \arrow[rr] \arrow[dr] & \arrow[d, phantom, "\circlearrowleft"] & B \\ & C_{k,n} \arrow[ur] & \end{tikzcd}$$

Moreover, $\mathrm{res} : CM(B) \xhookrightarrow{f.f.} CM(C)$.

Def $\mathcal{P} = \{ I \in \binom{n}{k} : M_I \in CM(B) \}$ is a (connected) positroid.

2) Positroid varieties

$Gr_{k,n} = \{ V \leq \mathbb{C}^n : \dim V = k \}$ Grassmannian ⇝ proj. variety.

$\mathbb{C}[\widehat{Gr}_{k,n}] = \mathbb{C}[\Delta_I : I \in \binom{n}{k}] / (\text{Plücker relations})$

↳ Plücker coordinate

$\Pi_{\mathcal{P}} = \{ V \in Gr_{k,n} : \Delta_I(V) = 0 \ \forall I \notin \mathcal{P} \}$ (closed) positroid variety.

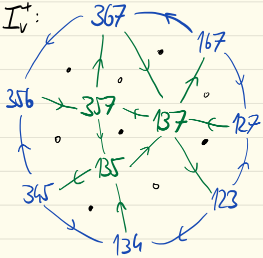
$v \in Q_0 ⇝ T_v^+ = eAe_v$, $T_v^- = (e_vAe)^\vee \in CM(B) \subseteq CM(C)$

$(-)^\vee = \mathrm{Hom}_Z(-, Z)$.

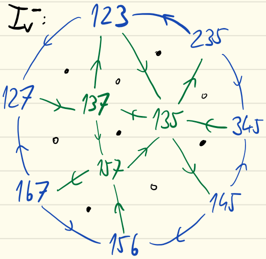
Consistency $\overset{[\text{ÇKP}]}{\Longrightarrow}$ rk $T_v^\pm = 1 \overset{[\text{JKS}]}{\Longrightarrow} T_v^\pm \cong M(I_v^\pm)$

⇝ two labellings of Q_0 by elements of $\binom{n}{k}$.

I_v^+:



I_v^-:



$\mathcal{F}^\pm = \{ I_v^\pm : v \in F_0 = \partial Q_0 \}$

$\Pi_P^\circ = \{ V \in \Pi_P : \Delta_I(V) \neq 0 \ \forall I \in \mathcal{F}^\pm \}$
open positroid variety

A_Q = cluster alg. associated to (Q, F), invertible frozen vars.

Thm (Galashin-Lam) Two isomorphisms

$$\eta^\pm : A_Q \xrightarrow{\sim} \mathbb{C}[\hat{\Pi}_P^\circ]. \quad \eta^\pm(x_v) = \Delta(I_v^\pm).$$

Upshot Two cluster algebra structures on $\mathbb{C}[\hat{\Pi}_P^\circ]$.

Rem Special case $P = \binom{n}{k} \Rightarrow \Pi_P = Gr_{k,n}$.
Π_P° = big cell (dense).

In (only) this case, cluster structures $\eta^\pm$ agree (Scott '06).

Conj (Muller-Speyer '17) $\eta^\pm$ quasi-coincide:

1) $\forall f \in A_Q$ frozen var., $\exists p$ Laurent mon. in frozens with
$$\eta^-(f) = \eta^+(p)$$
and $\forall x \in A_Q$ clust var. $\exists x'$ clust. var., q Laurent in frozens:
$$\eta^-(x) = \eta^+(x'q)$$

2) $x \longmapsto x'$ permutation of clust. vars, respecting compatibility, mutation.
3) technical 'balancing' condition on monomials p, q.

Thm (P '23+) The conjecture is true.

Rem 1) Both conjecture and theorem apply also to disconnected case.
2) Independent proof by Casals-Le-Sherman-Bennett-Weng, using methods from symplectic geometry.

3) Proof strategy

$$\begin{aligned} \mathrm{gproj}\,CMB &= \{X \in CM(B) : \mathrm{Ext}^{>0}_B(X,B)=0\} \\ \mathrm{ginj}\,CMB &= \{X \in CM(B) : \mathrm{Ext}^{>0}_B(B^\vee, X)=0\} \end{aligned} \subseteq CM(B)$$

Thm (P '22)

1) Both are stably 2-CY Frobenius exact categories.

2) $T^+ = eA \in \mathrm{gproj}\,CMB$, $T^- = (Ae)^\vee \in \mathrm{ginj}\,CMB$ are cluster-tilting.

3) $\mathrm{End}_B(T^+)^{op} \cong A \cong \mathrm{End}_B(T^-)^{op}$.

[Fu-Keller] $\Rightarrow$ clust. characters

$$\begin{tikzcd} \psi^+ : \mathrm{gproj}\,CM(B) \arrow[r] & A_Q \arrow[dr, "\eta^+", "\sim"'] & \\ & & \mathbb{C}[\hat{\Pi}^\circ_p] \\ \psi^- : \mathrm{ginj}\,CM(B) \arrow[r] & A_Q \arrow[ur, "\sim", "\eta^-"'] & \end{tikzcd}$$

inducing bijections of reachable rigid objects with cluster monomials, ...

Prop (P) $\psi^\pm(M_I) = \Delta_I$ (when M_I in domain).

[Fraser-Keller] To prove the conjecture, find $\varphi : D^b(\mathrm{ginj}\,CMB) \longrightarrow D^b(\mathrm{gproj}\,CMB)$ such that:

1)
$$\begin{tikzcd} K^b(\mathrm{add}\,T^-) \arrow[r] \arrow[d, "\tilde{\varphi}"'] \arrow[dr, phantom, "\circlearrowleft"] & D^b(\mathrm{ginj}\,CMB) \arrow[r] \arrow[d, "\varphi"] \arrow[dr, phantom, "\circlearrowleft"] & \underline{\mathrm{ginj}}\,CM(B) \arrow[d, "\underline{\varphi}"] \\ K^b(\mathrm{add}\,T^+) \arrow[r] & D^b(\mathrm{gproj}\,CMB) \arrow[r] & \underline{\mathrm{gproj}}\,CM(B) \end{tikzcd}$$

2)
$$\begin{tikzcd} \mathrm{add}\,T^- \arrow[r, "\psi^-"] \arrow[d, "\varphi"] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{C}[\hat{\Pi}^\circ_p] \arrow[d, equal] \\ D^b(\mathrm{gproj}\,CMB) \arrow[r, "\psi^+"] & \mathbb{C}[\hat{\Pi}^\circ_p] \end{tikzcd}$$

3) $\underline{\varphi}$ is an equivalence

4) $\underline{\varphi}\,T^- \overset{mut}{\sim} T^+$

Sketch Proof $\varphi = D^b(\mathrm{ginj}\,CMB) \xleftrightarrow{\sim} D^b(\mathrm{mod}\,B) \xleftrightarrow{\sim} D^b(\mathrm{gproj}\,CMB)$

$\rightsquigarrow$ $\underline{\varphi} = \underline{\mathrm{ginj}}\,CMB \xrightarrow{\sim} D_{sg}(B) \xleftarrow{\sim} \underline{\mathrm{gproj}}\,CMB \Longrightarrow (3)$.

Main step Show $T^+ = \Omega^2 T^- \in D_{sg}(B)$

(g-vector calculation, using ideas from CKP)

$\Rightarrow$ (4) by cluster theory (Ford-Serhiyenko)

$\Rightarrow$ (2) by geometry (Muller-Speyer + CKP)

1) $0 \to \underbrace{T^+ \to P_1 \to P_0}_{\xi} \to T^- \to 0$

$F = \mathrm{Hom}(T^+, -) : K^b(\mathrm{add}\,T^+) \xrightarrow{\sim} K^b(\mathrm{proj}\,A) = D^b(A)$ (P '22)

[ÇKP+e]: in $D^b(A)$, $F\xi \cong A^\vee$ is tilting.

$\mathrm{End}_A(A^\vee)^{op} = A$, so:

$$\begin{tikzcd}
T^- \arrow[d, mapsto] & K^b(\mathrm{add}\,T^-) \arrow[d, "\tilde{\varphi}"] \arrow[dr, "T^- \mapsto T^-"] & \\
\xi & K^b(\mathrm{add}\,T^+) \arrow[r, "\xi \mapsto \xi \cong T^-"'] \arrow[ur, phantom, "\circlearrowleft", near end] & D^b(\mathrm{mod}\,B)
\end{tikzcd}$$

$\square$

Rem In practice: take $X \in \mathrm{ginj}\,CM(B)$ reachable rigid, i.e. $\psi^-(X) \in \mathbb{C}[\Pi^{\circ}_{J}]$ is a cluster monomial for η^-.

$$\begin{tikzcd}
0 \arrow[r] & \overset{\overset{\mathrm{gproj}\,CMB}{\rotatebox{90}{$\in$}}}{\Omega X} \arrow[r] & P \arrow[r, "\text{proj cover}"] & X \arrow[r] & 0 \\
0 \arrow[r] & \Omega X \arrow[r, "\text{left proj}\,B\ \text{approx}"'] \arrow[u, equal] & Q \arrow[r] \arrow[u] & \underset{\underset{\mathrm{gproj}\,CMB.}{\rotatebox{90}{$\in$}}}{X'} \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

Then $\psi^-(X) = \psi^+(X')\dfrac{\psi^+(P)}{\psi^+(Q)} \in \mathbb{C}[\hat{\Pi}^{\circ}_{J}]$.