The geometry and representation theory of frieze patterns

Matthew Pressland

University of Glasgow

Durham, 25.03.2024

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The geometry and representation theory of ~~frieze patterns~~ SL_2-tilings

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Knitting vertically

Starting from a quiddity sequence, we can build an SL_2-tiling by computing downwards...

$$
\begin{array}{ccccccccccccccccccccccc}
 & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \\
\cdots & & & 1 & & 4 & & 1 & & 3 & & 2 & & 1 & & 4 & & 2 & & 1 & & & \cdots
\end{array}
$$

Knitting vertically

Starting from a quiddity sequence, we can build an SL_2-tiling by computing downwards...

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\begin{array}{ccccccccccccccccccccccc}
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\cdots & & & 1 & & 4 & & 1 & & 3 & & 2 & & 1 & & 4 & & 2 & & 1 & & & \cdots \\
 & & 1 & & 3 & & 3 & & 2 & & 5 & & 1 & & 3 & & 7 & & 1 & & 3 & &
\end{array}
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Knitting vertically

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 & 1 & & 3 & & 3 & & 2 & & 5 & & 1 & & 3 & & 7 & & 1 & & 3 & \\
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\cdots & & 2 & & 2 & & 5 & & 3 & & 2 & & 2 & & 5 & & 3 & & 2 & & \cdots \\
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 & 1 & & 3 & & 3 & & 2 & & 5 & & 1 & & 3 & & 7 & & 1 & & 3 & & \\
\cdots & & 2 & & 2 & & 5 & & 3 & & 2 & & 2 & & 5 & & 3 & & 2 & & \cdots \\
 & 5 & & 1 & & 3 & & 7 & & 1 & & 3 & & 3 & & 2 & & 5 & & 1 & & \\
\cdots & & 2 & & 1 & & 4 & & 2 & & 1 & & 4 & & 1 & & 3 & & 2 & & \cdots
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\cdots & & 1 & & 4 & & 1 & & 3 & & 2 & & 1 & & 4 & & 2 & & 1 & & \cdots \\
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\cdots & & 2 & & 2 & & 5 & & 3 & & 2 & & 2 & & 5 & & 3 & & 2 & & \cdots \\
 & 5 & & 1 & & 3 & & 7 & & 1 & & 3 & & 3 & & 2 & & 5 & & 1 & \\
\cdots & & 2 & & 1 & & 4 & & 2 & & 1 & & 4 & & 1 & & 3 & & 2 & & \cdots \\
 & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} &
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\cdots & & 2 & & 2 & & 5 & & 3 & & 2 & & 2 & & 5 & & 3 & & 2 & & \cdots \\
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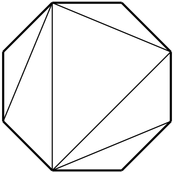
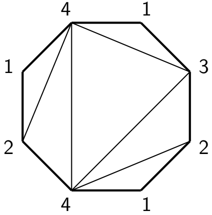
\cdots & & 2 & & 1 & & 4 & & 2 & & 1 & & 4 & & 1 & & 3 & & 2 & & \cdots \\
 & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} &
\end{array}
$$



Knitting vertically

Starting from a quiddity sequence, we can build an SL_2-tiling by computing downwards...

	1		**1**		**1**		**1**		**1**		**1**		**1**		**1**		**1**		**1**	
···		1		4		1		3		2		1		4		2		1		···
	1		3		3		2		5		1		3		7		1		3	
···		2		2		5		3		2		2		5		3		2		···
	5		1		3		7		1		3		3		2		5		1	
···		2		1		4		2		1		4		1		3		2		···
	1		**1**		**1**		**1**		**1**		**1**		**1**		**1**		**1**		**1**	



Knitting horizontally

…or we can compute horizontally from a zig-zag of 1s.

1 1 1 1 1 1 1 1 1 1

1

1

1

1

1

1

1 1 1 1 1 1 1 1 1 1 1

Knitting horizontally

...or we can compute horizontally from a zig-zag of 1s.

	1		**1**		**1**		**1**		**1**		**1**		**1**		**1**		**1**		**1**	
		1		2																
			1																	
		1																		
			1																	
				1																
					1															
1		**1**		**1**		**1**		**1**		**1**		**1**		**1**		**1**		**1**		**1**

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...or we can compute horizontally from a zig-zag of 1s.

	1		**1**		**1**		**1**		**1**		**1**		**1**		**1**		**1**		**1**	
		1		2																
			1																	
		1		2																
			1																	
				1																
					1															
1		**1**		**1**		**1**		**1**		**1**		**1**		**1**		**1**		**1**		**1**

Knitting horizontally

…or we can compute horizontally from a zig-zag of 1s.

	1		**1**		**1**		**1**		**1**		**1**		**1**		**1**		**1**		**1**	
		1		2																
			1		5															
		1		2																
			1																	
				1																
					1															
1		**1**		**1**		**1**		**1**		**1**		**1**		**1**		**1**		**1**		**1**

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	1		**1**		**1**		**1**		**1**		**1**		**1**		**1**		**1**		**1**	
		1		2																
			1		5															
		1		2																
			1		3															
				1																
					1															
1		**1**		**1**		**1**		**1**		**1**		**1**		**1**		**1**		**1**		**1**

Knitting horizontally

...or we can compute horizontally from a zig-zag of 1s.

1 1 1 1 1 1 1 1 1 1

1 2 3

1 5

1 2

1 3

1

1

1 1 1 1 1 1 1 1 1 1 1

Knitting horizontally

...or we can compute horizontally from a zig-zag of 1s.

	1		**1**		**1**		**1**		**1**		**1**		**1**		**1**		**1**		**1**	
		1		2		3														
			1		5															
		1		2		8														
			1		3															
				1																
					1															
1		**1**		**1**		**1**		**1**		**1**		**1**		**1**		**1**		**1**		**1**

Knitting horizontally

...or we can compute horizontally from a zig-zag of 1s.

	1		**1**		**1**		**1**		**1**		**1**		**1**		**1**		**1**		**1**	
		1		2		3														
			1		5		5													
		1		2		8														
			1		3															
				1																
					1															
1		**1**		**1**		**1**		**1**		**1**		**1**		**1**		**1**		**1**		**1**

Knitting horizontally

...or we can compute horizontally from a zig-zag of 1s.

···		**1**		**1**		**1**		**1**		**1**		**1**		**1**		**1**		**1**		**1**		···
	3		1		2		3		2		2		2		1		5		3		1	
···		2		1		5		5		3		3		1		4		14		2		···
	9		1		2		8		7		4		1		3		11		9		1	
···		4		1		3		11		9		1		2		8		7		4		···
	3		3		1		4		14		2		1		5		5		3		3	
···		2		2		1		5		3		1		2		3		2		2		···
	1		**1**		**1**		**1**		**1**		**1**		**1**		**1**		**1**		**1**		**1**	

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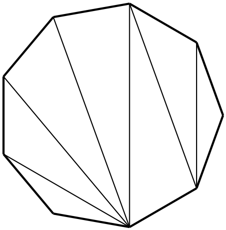
···		**1**		**1**		**1**		**1**		**1**		**1**		**1**		**1**		**1**		**1**		···
	3		1		2		3		2		2		2		1		5		3		1	
···		2		1		5		5		3		3		1		4		14		2		···
	9		1		2		8		7		4		1		3		11		9		1	
···		4		1		3		11		9		1		2		8		7		4		···
	3		3		1		4		14		2		1		5		5		3		3	
···		2		2		1		5		3		1		2		3		2		2		···
	1		**1**		**1**		**1**		**1**		**1**		**1**		**1**		**1**		**1**		**1**	



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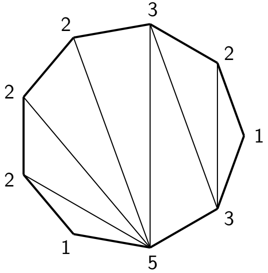
···		**1**		**1**		**1**		**1**		**1**		**1**		**1**		**1**		**1**		**1**		···
	3		1		2		3		2		2		2		1		5		3		1	
···		2		1		5		5		3		3		1		4		14		2		···
	9		1		2		8		7		4		1		3		11		9		1	
···		4		1		3		11		9		1		2		8		7		4		···
	3		3		1		4		14		2		1		5		5		3		3	
···		2		2		1		5		3		1		2		3		2		2		···
	1		**1**		**1**		**1**		**1**		**1**		**1**		**1**		**1**		**1**		**1**	



The Laurent phenomenon

This did not have to work!

$$\begin{array}{ccccccccccccc}
 & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \\
\cdots & & & x_1 & & \frac{1+x_2}{x_1} & & \frac{1+x_1}{x_2} & & x_2 & & \cdots & \\
 & & \frac{1+x_1}{x_2} & & x_2 & & \frac{1+x_1+x_2}{x_1x_2} & & x_1 & & \frac{1+x_2}{x_1} & & \\
\cdots & & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \cdots &
\end{array}$$

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 & & \frac{1+x_1}{x_2} & & x_2 & & \frac{1+x_1+x_2}{x_1x_2} & & x_1 & & \frac{1+x_2}{x_1} & & \\
\cdots & & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & & \cdots
\end{array}
$$

A sample calculation:

$$
\frac{1+\dfrac{1+x_1+x_2}{x_1x_2}}{\dfrac{1+x_2}{x_1}} = \frac{x_1(1+x_1+x_2+x_1x_2)}{x_1x_2(1+x_2)} = \frac{(1+x_1)(1+x_2)}{x_2(1+x_2)} = \frac{1+x_1}{x_2}
$$

This *Laurent phenomenon* implies we get integer values at $x_i = 1$.

The Laurent phenomenon

This did not have to work!

$$
\begin{array}{ccccccccccc}
 & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & \\
\cdots & & 1 & & 2 & & 2 & & 1 & & \cdots \\
 & 2 & & 1 & & 3 & & 1 & & 2 & \\
\cdots & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \cdots
\end{array}
$$

A sample calculation:

$$
\frac{1 + \dfrac{1 + x_1 + x_2}{x_1 x_2}}{\dfrac{1 + x_2}{x_1}} = \frac{x_1(1 + x_1 + x_2 + x_1 x_2)}{x_1 x_2 (1 + x_2)} = \frac{(1 + x_1)(1 + x_2)}{x_2(1 + x_2)} = \frac{1 + x_1}{x_2}
$$

This *Laurent phenomenon* implies we get integer values at $x_i = 1$.

Laurent phenomenon

Fomin–Zelevinsky define a *cluster algebra* $\mathcal{A}$ via recursively computed generators, called *cluster variables*, in $\mathbb{Q}(x_1, \ldots, x_n)$.

Theorem (Fomin–Zelevinsky '02)

Every cluster variable in $\mathcal{A}$ is a Laurent polynomial in $x_1, \ldots, x_n$.

Fomin–Zelevinsky's proof is combinatorial; Gross–Hacking–Keel give a conceptual, geometric proof.

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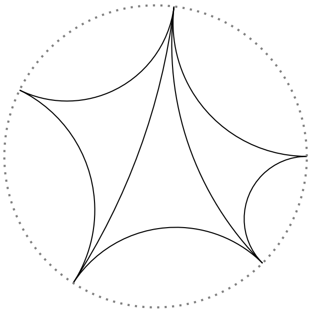
Observation (Caldero–Chapoton '06)

Given a frieze with n (interesting) rows, the formulae expressing arbitrary entries in terms of those in a zig-zag are given by cluster variables in a cluster algebra of type A_n.

$\Longrightarrow$ integrality, starting with a zig-zag of 1s.

Hyperbolic lengths

Given an ideal polygon in the Poincaré disc, we can measure the lengths of its sides and diagonals.



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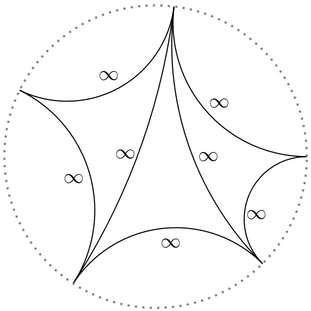
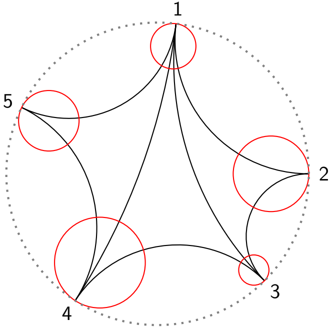


Hyperbolic lengths

Given an ideal polygon in the Poincaré disc, and a collection of horocycles at the cusps, we can measure the lambda lengths of its sides and diagonals.



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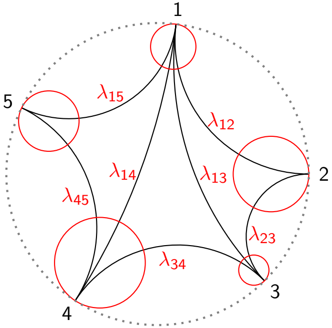

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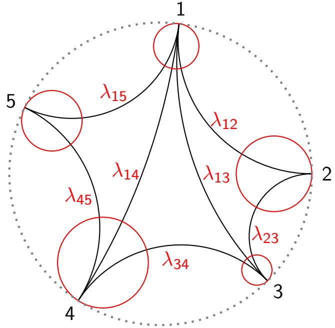
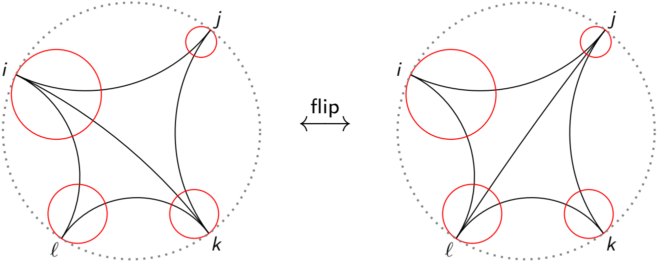


Decorated Teichmüller space $\widetilde{\mathcal{T}}_n$: moduli space of ideal n-gons in the Poincaré disc, with declared horocycles.

Flips

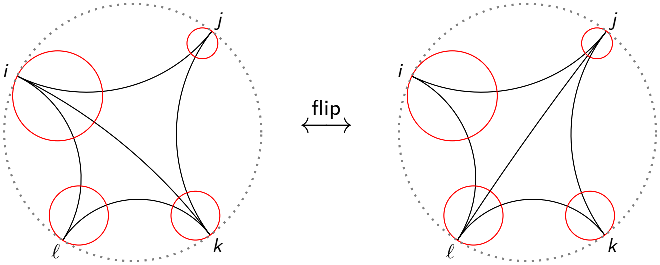
Whitehead move / Ptolemy relation:



$$\lambda_{ik}\lambda_{j\ell} = \lambda_{ij}\lambda_{k\ell} + \lambda_{i\ell}\lambda_{jk}$$

Flips

Whitehead move / Ptolemy relation:

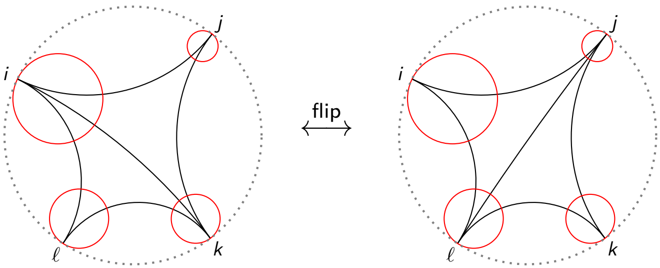


$$\lambda_{ik}\lambda_{j\ell} = \lambda_{ij}\lambda_{k\ell} + \lambda_{i\ell}\lambda_{jk}$$

Flip graph is connected: lambda lengths of arcs in a triangulation determine all others.

Flips

Whitehead move / Ptolemy relation:



$$\lambda_{ik}\lambda_{j\ell} = \lambda_{ij}\lambda_{k\ell} + \lambda_{i\ell}\lambda_{jk}$$

Flip graph is connected: lambda lengths of arcs in a triangulation determine all others.

Theorem (Penner, '87)

Each triangulation of the n-gon determines an isomorphism $\lambda\colon \widetilde{\mathcal{T}}_n \xrightarrow{\sim} \mathbb{R}^{2n-3}_{>0}$.

Back to SL_2-tilings

The lambda lengths of an ideal *n*-gon
fit into an SL_2-tiling (with coefficients).

$$
\begin{array}{ccccccccccccccccccccc}
 & \boldsymbol{\lambda_{12}} & & \boldsymbol{\lambda_{23}} & & \boldsymbol{\lambda_{34}} & & \boldsymbol{\lambda_{45}} & & \boldsymbol{\lambda_{56}} & & \boldsymbol{\lambda_{67}} & & \boldsymbol{\lambda_{78}} & & \boldsymbol{\lambda_{18}} & & \boldsymbol{\lambda_{12}} & & \boldsymbol{\lambda_{23}} & \\
\cdots & & \lambda_{13} & & \lambda_{24} & & \lambda_{35} & & \lambda_{46} & & \lambda_{57} & & \lambda_{68} & & \lambda_{17} & & \lambda_{28} & & \lambda_{13} & & \cdots \\
 & \lambda_{38} & & \lambda_{14} & & \lambda_{25} & & \lambda_{36} & & \lambda_{47} & & \lambda_{58} & & \lambda_{16} & & \lambda_{27} & & \lambda_{38} & & \lambda_{14} & \\
\cdots & & \lambda_{48} & & \lambda_{15} & & \lambda_{26} & & \lambda_{37} & & \lambda_{48} & & \lambda_{15} & & \lambda_{26} & & \lambda_{37} & & \lambda_{48} & & \cdots \\
 & \lambda_{47} & & \lambda_{58} & & \lambda_{16} & & \lambda_{27} & & \lambda_{38} & & \lambda_{14} & & \lambda_{25} & & \lambda_{36} & & \lambda_{47} & & \lambda_{58} & \\
\cdots & & \lambda_{57} & & \lambda_{68} & & \lambda_{17} & & \lambda_{28} & & \lambda_{13} & & \lambda_{24} & & \lambda_{35} & & \lambda_{46} & & \lambda_{57} & & \cdots \\
 & \boldsymbol{\lambda_{56}} & & \boldsymbol{\lambda_{67}} & & \boldsymbol{\lambda_{78}} & & \boldsymbol{\lambda_{18}} & & \boldsymbol{\lambda_{12}} & & \boldsymbol{\lambda_{23}} & & \boldsymbol{\lambda_{34}} & & \boldsymbol{\lambda_{45}} & & \boldsymbol{\lambda_{56}} & & \boldsymbol{\lambda_{67}} &
\end{array}
$$

Back to SL_2-tilings

The lambda lengths of an ideal *n*-gon
fit into an SL_2-tiling (with coefficients).

$$
\begin{array}{ccccccccccccccccccccc}
 & \boldsymbol{\lambda_{12}} & & \boldsymbol{\lambda_{23}} & & \boldsymbol{\lambda_{34}} & & \boldsymbol{\lambda_{45}} & & \boldsymbol{\lambda_{56}} & & \boldsymbol{\lambda_{67}} & & \boldsymbol{\lambda_{78}} & & \boldsymbol{\lambda_{18}} & & \boldsymbol{\lambda_{12}} & & \boldsymbol{\lambda_{23}} & \\
\cdots & & \lambda_{13} & & \lambda_{24} & & \lambda_{35} & & \lambda_{46} & & \lambda_{57} & & \lambda_{68} & & \lambda_{17} & & \lambda_{28} & & \lambda_{13} & & \cdots \\
 & \lambda_{38} & & \lambda_{14} & & \lambda_{25} & & \lambda_{36} & & \lambda_{47} & & \lambda_{58} & & \lambda_{16} & & \lambda_{27} & & \lambda_{38} & & \lambda_{14} & \\
\cdots & & \lambda_{48} & & \lambda_{15} & & \lambda_{26} & & \lambda_{37} & & \lambda_{48} & & \lambda_{15} & & \lambda_{26} & & \lambda_{37} & & \lambda_{48} & & \cdots \\
 & \lambda_{47} & & \lambda_{58} & & \lambda_{16} & & \lambda_{27} & & \lambda_{38} & & \lambda_{14} & & \lambda_{25} & & \lambda_{36} & & \lambda_{47} & & \lambda_{58} & \\
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 & \boldsymbol{\lambda_{56}} & & \boldsymbol{\lambda_{67}} & & \boldsymbol{\lambda_{78}} & & \boldsymbol{\lambda_{18}} & & \boldsymbol{\lambda_{12}} & & \boldsymbol{\lambda_{23}} & & \boldsymbol{\lambda_{34}} & & \boldsymbol{\lambda_{45}} & & \boldsymbol{\lambda_{56}} & & \boldsymbol{\lambda_{67}} &
\end{array}
$$

The SL_2-relations are Ptolemy relations:

$$\lambda_{i,j}\lambda_{i+1,j+1} = \lambda_{i,j+1}\lambda_{i+1,j} + \lambda_{i,i+1}\lambda_{j,j+1}$$

and these relations imply all others.

$\implies$ positivity, starting from a zig-zag of 1s.

Back to SL_2-tilings

The lambda lengths of an ideal *n*-gon with sides of length 1 fit into an SL_2-tiling.

$$\begin{array}{ccccccccccccccccccccc}
 & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & & \mathbf{1} & \\
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\end{array}$$

The SL_2-relations are Ptolemy relations:

$$\lambda_{i,j}\lambda_{i+1,j+1} = \lambda_{i,j+1}\lambda_{i+1,j} + 1$$

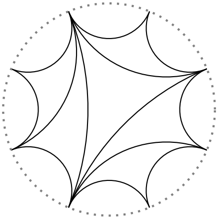
and these relations imply all others.

$\Longrightarrow$ positivity, starting from a zig-zag of 1s.

Cluster connections

Upshot: an SL_2-tiling of width n is an integer point of $\widetilde{\mathcal{T}}_{n+3}$.

	1		**1**		**1**		**1**		**1**		**1**		**1**		**1**		**1**		**1**	
···		1		4		1		3		2		1		4		2		1		···
	1		3		3		2		5		1		3		7		1		3	
···		2		2		5		3		2		2		5		3		2		···
	5		1		3		7		1		3		3		2		5		1	
···		2		1		4		2		1		4		1		3		2		···
	1		**1**		**1**		**1**		**1**		**1**		**1**		**1**		**1**		**1**	



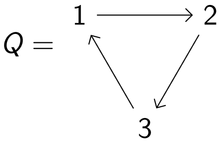
Cluster interpretation (Gekhtman–Shapiro–Vainshtein '05): $\widetilde{\mathcal{T}}_{n+3}$ is the positive part of a cluster variety of type A_n, defined over $\mathbb{C}$.

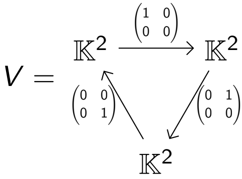
The same is true for $\mathrm{Gr}_{2,n}^{>0}$, the totally positive Grassmannian.

Quiver representations

A *quiver* Q is a directed graph (when it is being used to do algebra).

A *representation* V of the quiver is an assignment of a vector space to each vertex, and a linear map to each arrow.





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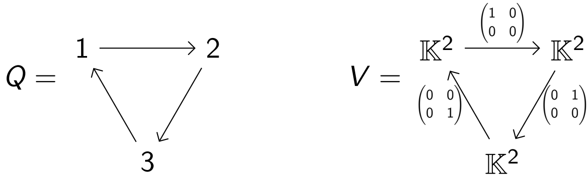
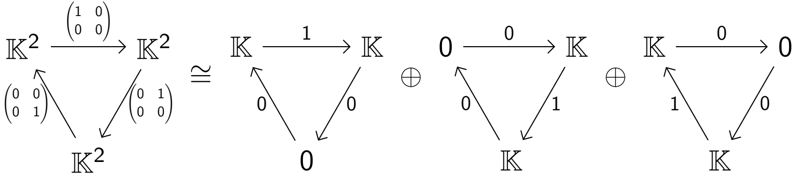


A representation is *indecomposable* if it is not a non-trivial direct sum.



Classification?

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Smith normal form:

$$Q = 1 \to 2 : \qquad V_r = \mathbb{K} \xrightarrow{1} \mathbb{K}, \quad V_n = \mathbb{K} \to 0, \quad V_c = 0 \to \mathbb{K}$$

Jordan normal form:

$$Q = \circlearrowleft * : \qquad V_{n,\lambda} = \overset{J_{n,\lambda}}{\circlearrowleft} * \quad \text{for } n \in \mathbb{N},\ \lambda \in \mathbb{K}$$

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Theorem (Gabriel)

A connected quiver Q has $< \infty$ indecomposable representations up to isomorphism if and only if it is an orientation of a simply-laced Dynkin diagram; indecomposables are in bijection with positive roots.

Type A_n: string diagrams

Indecomposable representations of A_n quivers can be drawn as string diagrams.

$$Q = 1 \to 2 \leftarrow 3 \to 4 \to 5$$

$$V = \mathbb{K} \xrightarrow{1} \mathbb{K} \xleftarrow{1} \mathbb{K} \xrightarrow{1} \mathbb{K} \xrightarrow{1} \mathbb{K} = \begin{smallmatrix} 1 & & 3 & & \\ & 2 & & 4 & \\ & & & & 5 \end{smallmatrix}$$

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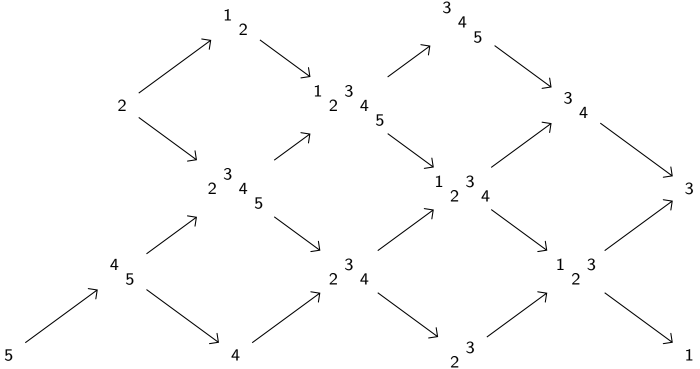
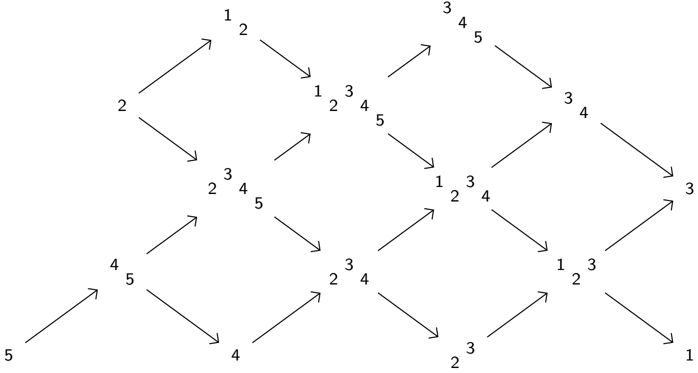
We can describe the entire category rep Q this way.



Counting subrepresentations

For each representation, count the number of subrepresentations (=down-closed subsets, viewing the string diagram as a poset).



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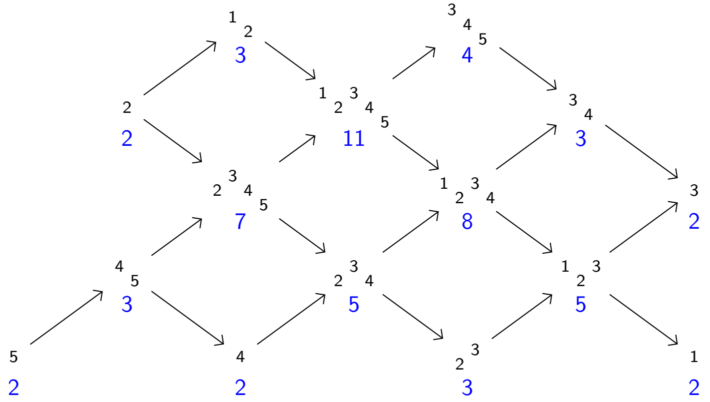

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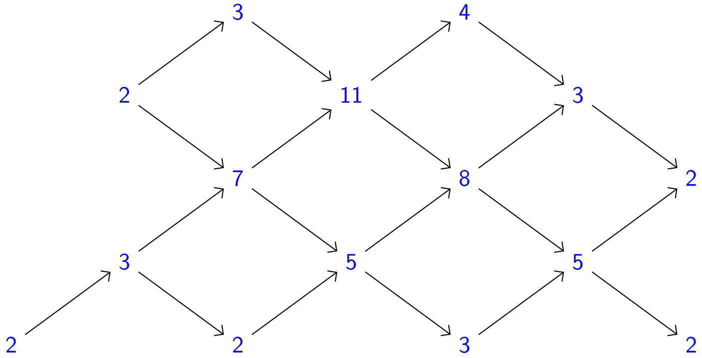

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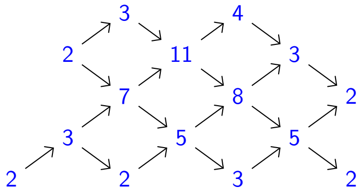

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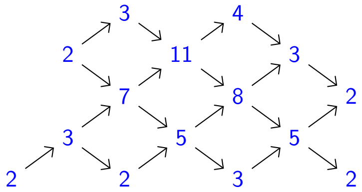

	1		**1**		**1**		**1**		**1**		**1**		**1**		**1**		**1**		**1**	
⋯		2		1		3		4		1		2		2		3		2		⋯
	5		1		2		11		3		1		3		5		5		1	
⋯		2		1		7		8		2		1		7		8		2		⋯
	3		1		3		5		5		1		2		11		3		1	
⋯		1		2		2		3		2		1		3		4		1		⋯
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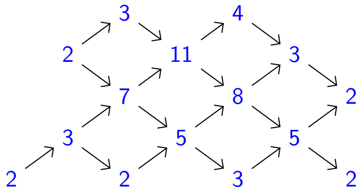

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	5		1		2		11		3		1		3		5		5		1	
⋯		2		1		7		8		2		1		7		8		2		⋯
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⋯		1		2		2		3		2		1		3		4		1		⋯
	1		**1**		**1**		**1**		**1**		**1**		**1**		**1**		**1**		**1**	

We found an SL_2-tiling!

The bounded derived category

For $V \in \operatorname{rep} Q$ and $i \in \mathbb{Z}$, introduce a formal symbol $\Sigma^i V$.

Objects of the *bounded derived category* $\mathcal{D}^{\mathrm{b}} Q$ are formal direct sums of these symbols.

Morphisms in $\mathcal{D}^{\mathrm{b}} Q$ are morphisms and extensions from $\operatorname{rep} Q$:

$$\operatorname{Hom}_{\mathcal{D}^{\mathrm{b}} Q}(\Sigma^i V, \Sigma^j W) = \operatorname{Ext}_Q^{j-i}(V, W).$$

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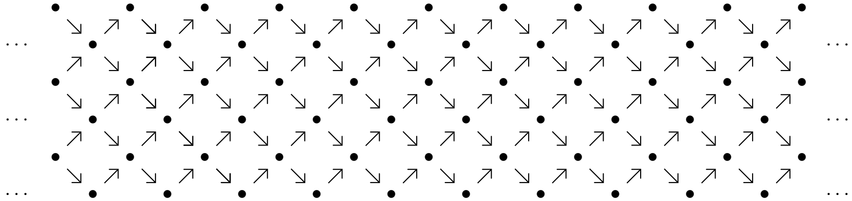

Symmetries

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On morphisms, Σ is the identity.

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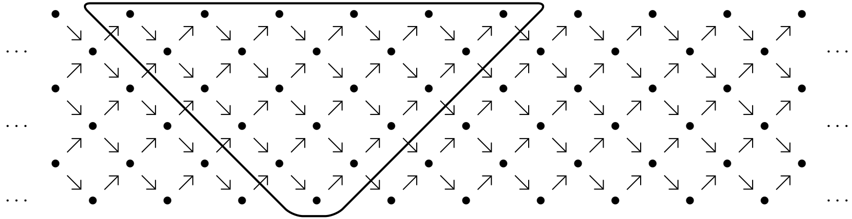
In type A, the autoequivalence Σ is a glide reflection, with rep Q as a fundamental domain.



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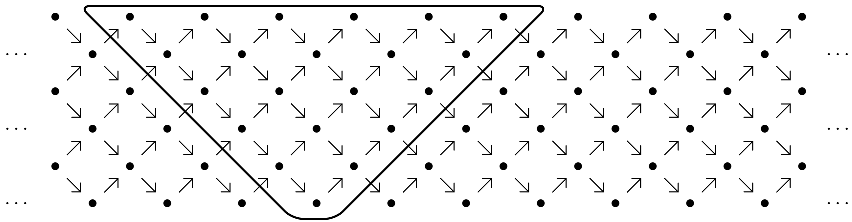
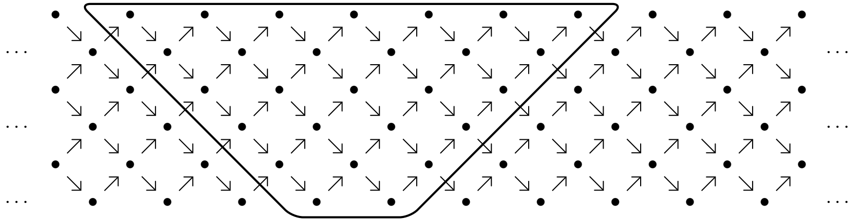


A second autoequivalence, τ, acts by translation to the left.

Orbit category

The symmetry $\Sigma^{-1} \circ \tau$ is the glide symmetry of an SL_2-tiling.



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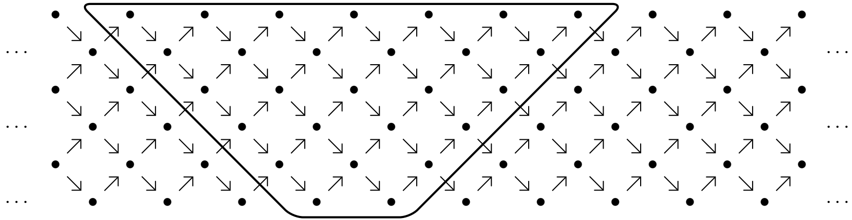

Definition (Buan–Marsh–Reineke–Reiten–Todorov)

For an acyclic quiver Q, the *cluster category* $\mathcal{C}_Q$ is the orbit category

$$\mathcal{C}_Q := \mathcal{D}^{\mathrm{b}} Q / (\Sigma^{-1} \circ \tau).$$

Same objects as $\mathcal{D}^{\mathrm{b}} Q$, morphisms

$$\mathrm{Hom}_{\mathcal{C}_Q}(X, Y) = \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}_{\mathcal{D}^{\mathrm{b}} Q}(X, (\Sigma^{-1} \circ \tau)^n Y).$$

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Remark

See also Caldero–Chapoton–Schiffler for type A.
See also Amiot for non-acyclic quivers.
Many further generalisations: Plamondon, Geiß–Leclerc–Schröer, Buan–Iyama–Reiten–Scott, Jensen–King–Su, Demonet–Iyama, P, Wu, Keller–Wu,...

Cluster character

The Caldero–Chapoton cluster character formula

$$\mathrm{CC}(X) = x^{\mathrm{ind}\, X} \sum_{e \leqslant \underline{\dim}\, GX} \chi(\mathrm{Gr}_e(GX)) x^{-B \cdot e}$$

computes cluster variables (expressed in a chosen initial cluster) from (reachable, rigid) indecomposable objects of $\mathcal{C}_Q$.

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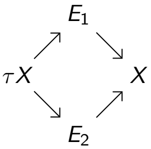
Key fact: for a triangle $\tau X \to \bigoplus_{i=1}^{k} E_i \to X$, we have

$$\mathrm{CC}(X)\,\mathrm{CC}(\tau X) = \prod_{i=1}^{k} \mathrm{CC}(E_i) + 1$$



$\Longrightarrow$ SL_2-relation!

SL_2-tiling on a cluster category

At $x \equiv 1$, we have

$$\mathsf{CC}(X) = \sum_{e \leqslant \underline{\dim}\, GX} \chi(\mathsf{Gr}_e(GX)),$$

which is a (weighted) sum of subrepresentations of GX.

For Q of type A_n and X indecomposable, we even have

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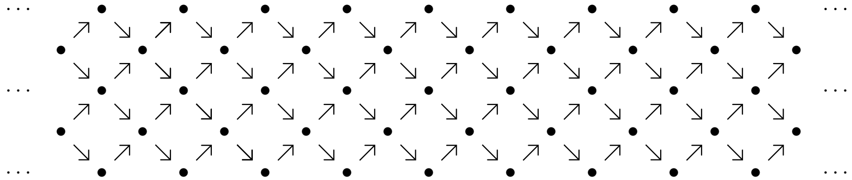

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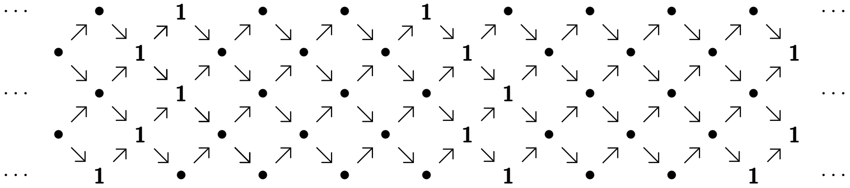

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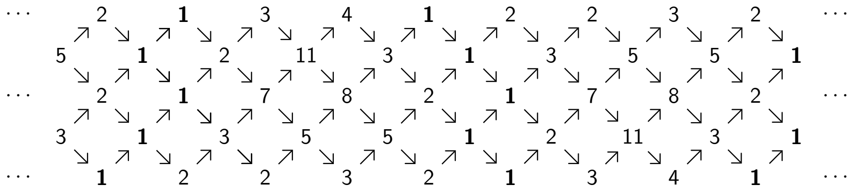

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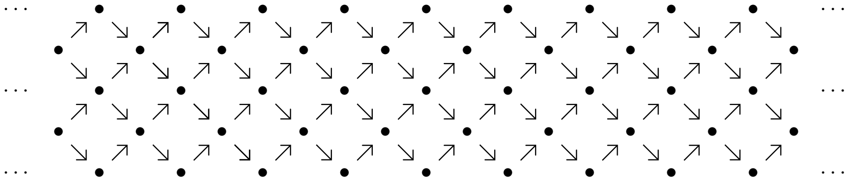

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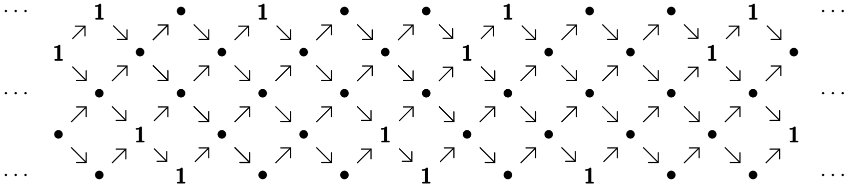

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