Perfect matching modules for dimer algebras

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Setting

- Fix integers $1 \leq k < n$. We study the Grassmannian G_k^n of k-subspaces of $\mathbb{C}^n$, and the coordinate ring $\mathbb{C}[\hat{G}_k^n]$ of its affine cone.
- The 'standard' generators of $\mathbb{C}[\hat{G}_k^n]$ are Plücker coordinates Δ_I for $I \in \binom{n}{k} = \{I \subseteq \{1, \ldots, n\} : |I| = k\}$.
- By work of Scott, $\mathbb{C}[\hat{G}_k^n]$ has a cluster algebra structure, in which all Δ_I are cluster variables.
- This cluster algebra is categorified by Jensen–King–Su and Baur–King–Marsh, via certain bipartite graphs called dimer models.
- Aim: compare two formulae for computing (particular) cluster monomials in this cluster algebra, one combinatorial and one representation-theoretic.

- Take a disc with marked points $1, \ldots, n$ on its boundary.
- A dimer D is a bipartite graph in the interior of the disc, together with n 'half-edges' connecting black nodes to these marked points.
- $k = \#\{\text{black nodes}\} - \#\{\text{white nodes}\}$.
- Each marked point is the source of a zig-zag path (turn right at black nodes, left at white nodes) leading to some other marked point. We insist that these paths satisfy some technical conditions, suppressed here, making them into a 'Postnikov diagram'.
- We also assume (mainly to make connections to existing literature) that the strand ending at marked point i starts and marked point $i + k$.

A labelling rule and a cluster of Plückers

- Label each face of D right of the zig-zag path ending at i by i. This process gives each face f a label $I(f) \in \binom{n}{k}$. We write $\mathcal{C}(D)$ for the set of all these face labels.

- This process determines a cluster $\{\Delta_I : I \in \mathcal{C}(D)\}$ of Plücker coordinates in the Grassmannian cluster algebra.

- We work in the Laurent polynomial ring over this cluster, and for $w \in \mathbb{Z}^{\mathcal{C}(D)}$ write
$$\Delta^w = \prod_{J \in \mathcal{C}(D)} \Delta_J^{w_J}.$$

The Marsh–Scott partition function

- A perfect matching μ of D is a set of edges of D (including half-edges) such that every node of D is incident with exactly one edge of μ.
- Since D has exactly k more black vertices than white, any perfect matching μ must include exactly k half-edges, and the so the boundary marked points adjacent to these half-edges form a set $\partial\mu \in \binom{n}{k}$.

Theorem (Marsh–Scott)

For each $I \in \binom{n}{k}$,

$$\sum_{\mu:\partial\mu=I} \Delta^{wt(\mu)} = \overleftarrow{\Delta}_I.$$

Here $\overleftarrow{\Delta}_I$ is the 'twisted Plücker coordinate' labelled by I—this is one of the cluster monomials—and $wt(\mu) \in \mathbb{Z}^{\mathcal{C}(D)}$ is a vector of weights attached combinatorially to a perfect matching.

The JKS category

- D also determines an algebra A_D by taking the dual quiver, with relations $p_\alpha^+ = p_\alpha^-$ whenever there are paths p_α^+ and p_α^- completing an arrow α to a cycle around a black $(+)$ and white $(-)$ node.
- A_D is free of finite rank over a central subalgebra $Z \cong \mathbb{C}[[t]]$.
- Let e be the sum of vertex idempotents at the boundary tiles, and $B = eA_De$; this algebra is also free of finite rank over Z.

Theorem (Jensen–King–Su)

The category $\mathrm{CM}(B)$, of B-modules free of finite rank over Z, categorifies the cluster algebra $\mathbb{C}[\hat{G}_k^n]$. In particular, there is a bijection between isoclasses of (reachable) rigid objects of $\mathrm{CM}(B)$ and cluster monomials.

Theorem (Baur–King–Marsh)

The algebra B is independent of D, up to isomorphism. The B-module $T_D := eA_D$ is a maximal rigid object in $\mathrm{CM}(B)$, and $\mathrm{End}_B(T_D)^{\mathrm{op}} \cong A_D$.

The CC formula

- Fix a dimer model D, with corresponding maximal rigid object $T_D \in \mathrm{CM}(B)$.

- Let $F = \mathrm{Hom}_B(T_D, -)$ and $G = \mathrm{Ext}^1_B(T_D, -)$; both are functors $\mathrm{CM}(B) \to \mathrm{mod}\, A_D$.

- Then the Caldero–Chapoton map (which gives the JKS bijection) is

$$\mathrm{CC}(X) = \sum_{N \leq GX} \Delta^{wt(N)} \qquad \left(\text{cf. MS: } \overleftarrow{\Delta}_I = \sum_{\mu : \partial\mu = I} \Delta^{wt(\mu)}\right)$$

- Here $wt(N) \in \mathbb{Z}^{\mathcal{C}(D)}$ is computed from projective resolutions of the A_D-modules FX and N.

MS=CC

- For each $I \in \binom{n}{k}$, JKS describe explicitly a module $M_I \in \mathrm{CM}(B)$.
- Each M_I has a 'canonical' projective cover P_I, yielding an exact sequence

$$0 \longrightarrow \Omega M_I \longrightarrow P_I \longrightarrow M_I \longrightarrow 0,$$

Proposition (Çanakçı–King–P)

$\overleftarrow{\Delta}_I = \mathrm{CC}(\Omega M_I)$.

- Together with the MS formula, this gives us

$$\sum_{\mu:\partial\mu=I} \Delta^{wt(\mu)} = \overleftarrow{\Delta}_I = \sum_{N \leq G\Omega M_I} \Delta^{wt(N)}$$

- Aim: use equality of the outer terms to deduce representation-theoretic information about A_D and B.

Perfect matching modules

$$\sum_{\mu:\partial\mu=I} \Delta^{wt(\mu)} = \sum_{N \leq G\Omega M_I} \Delta^{wt(N)}$$

- Let μ be a perfect matching of D.
- Define an A_D-module $\hat{N}_\mu$ by placing a copy of $Z = \mathbb{C}[[t]]$ at each vertex, and having arrows act by multiplication with t if they are dual to edges in μ, and by the identity otherwise.
- Applying F to the exact sequence defining ΩM_I gives an exact sequence

$$FP_I \xrightarrow{f} FM_I \xrightarrow{g} G\Omega M_I \longrightarrow 0$$

Theorem (Çanakçı–King–P, 'MS=CC')

The submodules of FM_I containing $\operatorname{im} f$ are precisely the $\hat{N}_\mu$ with $\partial\mu = I$. Thus, setting $N_\mu := g\hat{N}_\mu$, the assignment $\mu \mapsto N_\mu$ is a bijection $\{\mu : \partial\mu = I\} \xrightarrow{\sim} \{N \leq G\Omega M_I\}$. Moreover, $wt(\mu) = wt(N_\mu)$.