Week 8: Cauchy's Multiplication Theorem

This week I'll attempt to explain the proof of Cauchy's Multiplication Theorem given in the lectures. The theorem states that if $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are absolutely convergent, then the series $\sum_{n=0}^{\infty} c_n$, where $c_n = \sum_{m=0}^{n} a_m b_{n-m}$, is absolutely convergent, and:

$$\sum_{n=0}^{\infty} c_n = \left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right)$$

Before beginning the proof, it may be worth trying some examples of multiplying finite sums, and see that a similar result holds (multiplying a polynomial in x by a polynomial in y would be a good example). The main idea in the proof is to take a limit of this process.

So to begin the proof, we take $N, K \in \mathbb{N}_0$, and choose $K \geq 2N$, for reasons that will become apparent later. We want a neat way of writing some of the infinite series that will appear later on, so define:

$$A = \sum_{n=0}^{\infty} a_n \qquad \overline{A} = \sum_{n=0}^{\infty} |a_n|$$
$$B = \sum_{n=0}^{\infty} b_n \qquad \overline{B} = \sum_{n=0}^{\infty} |b_n|$$

If we want to get the result in the theorem, we need to find that $\sum_{k=0}^{K} c_k$ gets close to the product $\left(\sum_{n=0}^{N} a_n\right)\left(\sum_{m=0}^{N} b_n\right)$ when N and K are large. So we bound:

$$\left|\sum_{k=0}^{K} c_k - \left(\sum_{n=0}^{N} a_n\right)\left(\sum_{m=0}^{N} b_m\right)\right| \leq \sum_{\substack{0\leq n\leq K \\ 0\leq m\leq K \\ n>N \text{ or } m>N}} |a_n||b_m|$$

This is perhaps not so easy to see, but if you stare at the definition of c_n, you should be able to see that $\sum_{k=0}^{K} c_k$ is going to be a sum of things that look like $a_m b_{k-m}$ for $k < K$, and see that these terms also occur in $\left(\sum_{n=0}^{N} a_n\right)\left(\sum_{m=0}^{N} b_n\right)$ when $m < N$ and $k - m < N$. So the only terms left after cancelling look like $a_i b_j$ with either $i > N$ or $j > N$, which (up to a change of notation) is what we have on the right-hand side, which now follows by applying the triangle inequality. This is also where we use $K \geq 2N$, so we don't end up subtracting any terms in the product $\left(\sum_{n=0}^{N} a_n\right)\left(\sum_{m=0}^{N} b_n\right)$ that don't actually appear in $\sum_{k=0}^{K} c_k$. We then notice that:

$$\sum_{\substack{0\leq n\leq K \\ 0\leq m\leq K \\ n>N \text{ or } m>N}} |a_n||b_m| \leq \left(\sum_{n=0}^{K} |a_n|\right)\left(\sum_{m=N}^{K} |b_m|\right) + \left(\sum_{m=0}^{K} |b_m|\right)\left(\sum_{n=N}^{K} |a_n|\right)$$

as all the terms are positive, and all the terms on the left-hand side certainly appear on the right-hand side. (I think this is actually an equality, but as we only need the inequality anyway I'm not going to worry about it too much).

Now we have some idea what's going on with the partial sums, we can start taking limits. We'd like to take K and N to infinity, but we don't really know how to do this for a number of reasons. Does the order matter? Does the requirement that $K \geq 2N$ affect how we can take the limit? These problems are somewhat complicated, so we'll take a more delicate approach. Taking $K \to \infty$ on the right-hand side only makes things bigger, so using our notation from above, we get the inequality:

$$\left| \sum_{k=0}^{K} c_k - \left(\sum_{n=0}^{N} a_n \right) \left(\sum_{m=0}^{N} b_n \right) \right| \leq \overline{A} \sum_{m=N}^{\infty} |b_m| + \overline{B} \sum_{n=N}^{\infty} |a_n| =: S_N$$

Note that as tails of convergent series go to zero, the limit of the right-hand side as $N \to \infty$ is zero. Note that we don't take this limit on the left-hand side because we don't quite know what to do with the K, which is bigger than $2N$ so has to go to infinity as well.

Now if we fix $\varepsilon > 0$, there is $N_1 \in \mathbb{N}$ such that $S_N < \frac{1}{2}\varepsilon$ for all $N \geq N_1$. By algebra of limits, we also know that $\left(\sum_{n=0}^{N} a_n\right)\left(\sum_{m=0}^{N} b_m\right) \to AB$ as $N \to \infty$, so we can find $N_2 \in \mathbb{N}$ such that:

$$\left| \left(\sum_{n=0}^{N} a_n \right) \left(\sum_{m=0}^{N} b_m \right) - AB \right| < \frac{1}{2}\varepsilon$$

for all $N \geq N_2$. Let $N_3 = \max\{N_1, N_2\}$. Then if $K \geq 2N_3$ (recall that we needed $K \geq 2N$ for all of the above calculations), we get from the triangle inequality that:

$$\begin{aligned} \left| \sum_{k=0}^{K} c_k - AB \right| &\leq \left| \sum_{k=0}^{K} c_k - \left(\sum_{n=0}^{N_3} a_n \right) \left(\sum_{m=0}^{N_3} b_m \right) \right| + \left| \left(\sum_{n=0}^{N_3} a_n \right) \left(\sum_{m=0}^{N_3} b_m \right) - AB \right| \\ &\leq \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon \end{aligned}$$

and hence $\sum_{k=0}^{K} c_k \to AB$ as $K \to \infty$. ("Splitting the difference" like this is a standard trick in analysis, so you should remember it).

As mentioned in the notes, the proof of absolute convergence is very similar, and follows by defining $\overline{c}_k = \sum_{n=0}^{k} |a_n||b_{k-n}|$, and noting that $|c_k| \leq \overline{c}_k$ for all k, so if we can show that $\sum_{k=0}^{\infty} \overline{c}_k$ converges, the comparison test will tell us that $\sum_{k=0}^{\infty} |c_k|$ converges. This can be shown by repeating the above proof with $\overline{c}_k$, $\overline{A}$ and $\overline{B}$ replacing c_k, A and B — it is probably a good idea to actually do this, making sure you understand all the steps.