Week 6: What Constants to Use

Many of the answers to Q2 on this week's problem sheet were vague about the constants involved, so I'll offer some advice on getting them right. The problem was to show that given a sequence $x_n \to \infty$ and a sequence $y_n \to \ell > 0$, the product sequence $x_n y_n$ diverges to infinity.

We should start thinking about a question like this by doing some rough work approximating the two sequences. This part needn't be written up in the final proof, but can be useful in constructing it. In this example, we notice that for any $C \in \mathbb{R}$ and $\varepsilon > 0$, then if n is big enough (we'll say how big later), we have $x_n > C$ and $|y_n - \ell| < \varepsilon$. So in particular:

$$x_n y_n > C(\ell - \varepsilon)$$

As we want to make $x_n y_n$ large, this seems helpful, provided that given any $M \in \mathbb{R}$, we can arrange for $C(\ell - \varepsilon) = M$. What we need is for C to be equal to $\frac{M}{\ell - \varepsilon}$, and (this is a bit more subtle) $\varepsilon < \ell$, else the inequality would be reversed in the above equation. So maybe we'll choose $\varepsilon = \ell/2$ and $C = 2M/\ell$.

Now we're ready to turn this into a proper proof. Let $M \in \mathbb{R}$. We know from the definitions that:

- There exists N_1 such that $x_n > 2M/\ell$ for all $n > N_1$.
- There exists N_2 such that $|y_n - \ell| < \ell/2$ for all $n > N_2$, or equivalently:

  $$\frac{\ell}{2} < y_n < \frac{3\ell}{2}$$

  for all $n > N_2$. (Alternatively you could use the inertia principle here to find N_2 such that $y_n > \ell/2$ for all $n > N_2$).

So let $N = \max\{N_1, N_2\}$. Then in particular, for all $n > N$:

$$x_n y_n > \frac{2M}{\ell} \times \frac{\ell}{2} = M$$

and hence $x_n y_n \to \infty$.

Note that the rough work that was used to decide that our constants should be $2M/\ell$ and $\ell/2$ is superfluous to the proof, and need not be included; once you've found constants that work, it doesn't matter how you found them.