Week 5: Finding the Tricks

As we found during the discussion in this week's class, the problem of showing that $n^{1/n}$ converges to 1 involves some slightly ingenious tricks. It is fairly natural to wonder how you're supposed to come up with these ideas, so this week I'll talk about that process a little bit, and hopefully demonstrate that it's not as hard to come up with a trick as it sometimes looks.

The first trick in this problem is given to us in the question, but I'll discuss it anyway — the idea is to write $n^{1/n} = 1 + h_n$. Why do this? Well, after (eventually) deciding that $n^{1/n} > 1$, and keeping in mind that we want to show that $n^{1/n}$ gets close to 1 as n goes to infinity, it is fairly natural to ask *how* much bigger $n^{1/n}$ is than 1 for any given n. Of course, this is given by $n^{1/n} - 1$, so we give this quantity the name h_n (and can now write $n^{1/n} = 1 + h_n$ as above). It is then clear that the problem of showing $n^{1/n} \to 1$ is equivalent to showing that $h_n \to 0$.

So how do we do this? Firstly, we should observe that:

$$(1 + h_n)^n = n$$

This is almost the only piece of information that we have about h_n, so it seems sensible to write it down (particularly as we are given the hint that the binomial theorem will be useful). We found in class that the appropriate trick here is to notice that all the terms in the binomial formula are positive, and just take the second one, so we get:

$$(1 + h_n)^n \geq \frac{n(n-1)}{2} h_n^2$$

Why we should do this is perhaps not obvious, so let's see how we might arrive at this conclusion by experimenting. One thing we might try first (as many of you did) is to use the binomial inequality, and get:

$$(1 + h_n)^n \geq 1 + nh_n$$

The problem with this is that if we rearrange to get a bound on h_n, we get:

$$h_n \leq \frac{(1 + h_n)^n - 1}{n} = \frac{n-1}{n} = 1 - \frac{1}{n} \to 1$$

But this is no good (we need to bound h_n above by something tending to 0), so how can we fix it? As some of you noticed, it would help to have extra ns on the denominator. So think about where the binomial inequality comes from — it is given by throwing away all the terms of the binomial expansion after the first two. So perhaps we will have more luck if we keep the third term; after all, it contains an n^2. Doing this instead, we get:

$$(1 + h_n)^n \geq 1 + nh_n + \frac{n(n-1)}{2} h_n^2$$

But this is also slightly unhelpful, as we now have a quadratic in h_n, so this is going to be difficult to re-arrange. So we need to get rid of one of the h_n terms. It shouldn't be the h_n^2 term, or we're just back where we were before, so throw away the nh_n term instead, and get:

$$(1+h_n)^n \geq 1 + \frac{n(n-1)}{2} h_n^2$$

Then we can rearrange and find:

$$h_n^2 \leq \frac{2\left((1+h_n)^n - 1\right)}{n(n-1)} = \frac{2n-1}{n(n-1)} = \frac{2-\frac{1}{n}}{n-1} \to 0$$

If we were being clever, we could also do what we did in class and throw the 1 away as well, as it does absolutely nothing useful and just gets in the way, but this is by no means essential. Either way, we get $h_n^2 \to 0$. The proof from this point follows a fairly standard form, so I'll leave it to you.

It's important to keep in mind that mathematical proof in general doesn't follow any kind of template. It's a creative process, and you may have to spend some time thinking about a problem or trying various approaches until it works.

A Warning About Algebra of Limits

It's important to be careful about which direction the implication in a theorem goes — getting it backwards can make things go badly wrong. Based on the idea today that involved using algebra of limits in the wrong direction, I give the following example of why you should be careful. Say I give you a sequence (x_n), and tell you that $x_n^2 \to 1$. You might be tempted to say that:

$$1 = \lim_{n\to\infty}(x_n^2) = (\lim_{n\to\infty} x_n)^2$$

by algebra of limits, so $\lim_{n\to\infty} x_n = \pm 1$. Unfortunately, this doesn't work, the problem being that the sequence x_n may not have a limit. For example, let $x_n = (-1)^n$. Then $x_n^2 = 1$ for all n, so $x_n^2 \to 1$, but x_n does not converge.

So have we found a counterexample to the algebra of limits theorem? Not at all — in this context, the theorem says that if $x_n \to \ell$, then $x_n^2 \to \ell^2$ (note the direction of the implication), and in our example, there is no such ℓ, so the hypothesis doesn't hold and the theorem doesn't apply.