Week 4: Completeness and Convergence

This week's theme is completeness. There are many ways of stating the completeness axiom, but the one you're using for the moment is that any increasing sequence that is bounded above converges. This allows us to prove convergence for a lot more sequences (but not actually find their limits, as that requires more work).

One particular class of sequences being discussed are recursive sequences (see problem 2), where we are given an initial value $x_1 = k$, and then define $x_n = f(x_{n-1})$ for $n \geq 2$, so each subsequent term is given by a function of the preceding one. Completeness is useful here because of the following observation; if x_n converges to x (and f is suitably "nice", but don't worry about that here, as in all the examples you see it will be), we have:

$$x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} f(x_{n-1}) = f(x)$$

and if we're lucky, we can solve for x to find the limit. However, this method only works if the sequence definitely converges, so we would like to be able to prove that, and completeness gives us a way to do it — if we can show that the sequence is increasing (by looking at the ratio or difference of successive terms) and bounded above (usually by induction), then we know the limit exists, and can find it by the previous argument. An equivalent version of completeness says we can do the same for decreasing sequences that are bounded below.

We can also (see problem 4) use the same methods to prove that a more general sequence converges, and get a bound on the limit. If a sequence x_n is increasing, and $x_n \leq k$ for all n (or at least all $n > N$ for some N — think about why this is true) then x_n converges to a limit x, with $x \leq k$. We can't say for sure what the limit is, but having bounds like this can be useful in some contexts (e.g. numerical analysis).

A good exercise (not directly related to the problem sheet) is to think about why the completeness axiom doesn't hold in $\mathbb{Q}$, i.e. come up with a sequence in $\mathbb{Q}$ that is increasing and bounded above, but does not converge. Hint: the completeness axiom for $\mathbb{R}$ should be more precisely stated "any increasing sequence *of real numbers* that is bounded above converges *to a real number*".