# Week 2: Proving Convergence

Many of you had trouble proving that a given sequence converged in this week's problem sheet. While the specifics of proving this will vary depending on the sequence given, the basic outline is normally roughly the same. So to give you an idea, I'll work through an example, and prove that the sequence:

$$a_n = \frac{3}{n^2} + \frac{4}{n^4}$$

tends to 0. First, look at the definition, to find out what we want to show. We require that for any $\varepsilon > 0$, there is some natural number $N \in \mathbb{N}$ such that for any $n > N$, $|a_n - 0| < \varepsilon$. Now for this particular sequence, $a_n > 0$, so $|a_n - 0| = a_n$, which is fairly helpful. So we want:

$$a_n = \frac{3}{n^2} + \frac{4}{n^4} < \varepsilon$$

and to do this, we need to make $n$ sufficiently large. Exactly how large isn't that easy to see, so we should approximate $a_n$ by a simpler sequence. Note that as $n^2 \leq n^4$ for all $n \geq 1$, we have $\frac{1}{n^4} \leq \frac{1}{n^2}$. So:

$$a_n \leq \frac{3}{n^2} + \frac{4}{n^2} = \frac{7}{n^2}$$

So if we can make $n$ large enough that $\frac{7}{n^2} < \varepsilon$, then $a_n < \varepsilon$ as well. So far so good. Now to find $N$, note that:

$$\frac{7}{n^2} < \varepsilon \iff n > \sqrt{\frac{7}{\varepsilon}}$$

So if we choose some $N \in \mathbb{N}$ bigger than $\sqrt{\frac{7}{\varepsilon}}$, then for any $n > N$, the value $\frac{7}{n^2}$ will be less than $\varepsilon$, and hence $a_n < \varepsilon$, which is what we wanted in the first place. So to conclude, for any $\varepsilon > 0$, and any $n > \sqrt{\frac{7}{\varepsilon}}$, we have that $|a_n - 0| < \varepsilon$. This is precisely the definition of $n \to 0$, so we have proved the result.

Note that if we want to show $a_n \to a$ for some $a \neq 0$, then we need to find $n$ such that $|a_n - a| < \varepsilon$, and so we should approximate $|a_n - a|$ by some sequence $b_n \geq |a_n - a|$ if necessary, rather than approximating $a_n$.